Wave Operators for Schrödinger Operators with Threshold Singularities, Revisited

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Abstract

The continuity property in the Sobolev space $W^{k,p}(\mathbb{R}^m)$ of wave operators of scattering theory for m-dimensional single-body Schrödinger operator is considered when the resolvent of the operator has singularities at the bottom of the continuous spectrum. It is shown that they are continuous in $W^{k,p}(\mathbb{R}^m)$, $0 \le k \le 2$ for $1 < p < 3$ but not for $p > 3$ if $m = 3$ and, for $1 < p < m/2$ but not for $p > m/2$ if $m \ge 5$. This extends downward the previously known interval of p for the continuity, $3/2 < p < 3$ for $m = 3$ and $m/(m-2) < p < m/2$ for $m \ge 5$. The formula which represents the integral kernel of the resolvent of the even dimensional free Schödinger operator as the superposition of exponential-polynomial like functions substantially simplifies the proof of the previous paper when $m \ge 6$ is even.

1 Introduction

Let $H_0 = -\Delta$ be the free Schrödinger operator with natural domain $D(H_0) = H^2(\mathbb{R}^m)$ and $H = H_0 + V(x)$ with real $V(x)$ such that

$$|V(x)| \le C\langle x\rangle^{-\delta} \quad \text{for some} \quad \delta > 2, \ \langle x\rangle = (1+|x|^2)^{\frac{1}{2}}. \tag{1.1}$$

We write $\mathcal{H} = L^2(\mathbb{R}^m)$ and $H^s(\mathbb{R}^m) = \{u \in \mathcal{H} \colon \partial^\alpha u \in \mathcal{H}, \ |\alpha| \le s\}$ is the Sobolev space, $s \ge 0$ being an integer. Then, H is selfadjoint in $\mathcal{H}$ with a core $C_0^\infty(\mathbb{R}^m)$ and it satisfies the following properties (see e.g. [14, 15, 17, 18, 19]):

(i) The spectrum $\sigma(H)$ of H consists of the absolutely continuous (AC for short) part $[0, \infty)$ and a finite number of non-positive eigenvalues $\{\lambda_j\}$ of finite multiplicities. The singular continuous spectrum and positive eigenvalues are absent from H.

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We write $\mathcal{H}_{ac}(H)$ for the AC spectral subspace of $\mathcal{H}$ for H and $P_{ac}(H)$ is the orthogonal projections onto $\mathcal{H}_{ac}(H)$.

(ii) Wave operators $W_\pm$ defined by strong limits

$$W_\pm u = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u, \quad u \in \mathcal{H} \tag{1.2}$$

exist and are complete: Image $W_\pm = \mathcal{H}_{ac}(H)$. They are unitary from $\mathcal{H}$ onto $\mathcal{H}_{ac}(H)$ and intertwine H_{ac} and H_0: For Borel functions f,

$$f(H)P_{ac}(H) = W_\pm f(H_0)W_\pm^*. \tag{1.3}$$

If follows that various mapping properties of $f(H)P_{ac}$ may be deduced from those of $f(H_0)$ if the corresponding ones of $W_\pm$ are known. In particular, if

$$W_\pm \in \mathbf{B}(L^p(\mathbb{R}^m)) \quad \text{for } p \in [p_1, p_2], \tag{1.4}$$

then $W_\pm^* \in \mathbf{B}(L^q(\mathbb{R}^m))$ for $q \in [q_2, q_1]$, $1/p_j + 1/q_j = 1$, $j = 1, 2$, and

$$\|f(H)P_{ac}(H)\|_{\mathbf{B}(L^q,L^p)} \le C_{pq}\|f(H_0)\|_{\mathbf{B}(L^q,L^p)}, \tag{1.5}$$

for these p and q with a constant C_{pq} which is *independent of f*. Here $f(H_0)$ is a Fourier multiplier by $f(\xi^2)$ and its mapping properties should be much easier to investigate than those of $f(H)P_{ac}(H)$. We define the Fourier transform $\hat{u}(\xi) = \mathcal{F}u(\xi)$ and the conjugate one $\mathcal{F}^*u(\xi)$ by

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^m} e^{-ix\xi}u(x)dx \text{ and } \mathcal{F}^*u(\xi) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{ix\xi}u(x)dx$$

respectively. The intertwining property (1.3) may be made more precise: Wave operators $W_\pm$ are transplantations ([20]) of the complete set of (generalized) eigenfunctions $\{e^{ix\xi} \colon \xi \in \mathbb{R}^m\}$ of $-\Delta$ by those of out-going and in-coming scattering eigenfunctions $\{\varphi_\pm(x,\xi) \colon \xi \in \mathbb{R}^m\}$ of $H = -\Delta + V$ ([15]):

$$W_\pm u(x) = \mathcal{F}_\pm^*\mathcal{F}u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \varphi_\pm(x,\xi)\hat{u}(\xi)d\xi,$$

where $\mathcal{F}_\pm$ and $\mathcal{F}_\pm^*$ are the generalized Fourier transforms associated with $\{\varphi_\pm(x,\xi) \colon \xi \in \mathbb{R}^m\}$ and the conjugate ones respectively defined by

$$\mathcal{F}_\pm u(\xi) = \int_{\mathbb{R}^d} \overline{\varphi_\pm(x,\xi)}u(x)dx, \quad \mathcal{F}_\pm^*u(\xi) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^d} \varphi_\pm(x,\xi)u(x)dx.$$

They satisfy the inversion formulas: $\mathcal{F}_\pm^*\mathcal{F}_\pm u = u$ for $u \in \mathcal{H}(H)$. It follows for multiplication operators M_F with Borel functions $F(\xi)$ on $L^2(\mathbb{R}^m)$ that

$$F(D_\pm) \equiv \mathcal{F}_\pm^* M_F \mathcal{F}_\pm u = W_\pm F(D)W_\pm^* u, \quad u \in \mathcal{H}_{ac}(H)$$

and the estimate (1.5) remains valid when $f(H)$ and $f(H_0)$ are replaced by $F(D_\pm)$ and $F(D)$.

In this paper we are interested in the problem that under what conditions $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$, which almost automatically implies that they are bounded also in $W^{k,p}(\mathbb{R}^m)$, $\leq k \leq 2$ (see below).

There is now a substantial literature on this problem ([4, 6, 7, 11, 22, 24, 27]) and it is known that the answer depends on the spectral properties of H at the bottom of the AC spectrum of H. We define

$$\mathcal{E} = \{u \in H^2(\mathbb{R}^m) \colon (-\Delta + V)u = 0\}, \tag{1.6}$$

the eigenspace of H associated with the eigenvalue 0 and

$$\mathcal{N} = \{u \in \langle x \rangle^s L^2(\mathbb{R}^m) : (1 + (-\Delta)^{-1}V)u = 0\} = 0. \tag{1.7}$$

The space $\mathcal{N}$ is independent of $1/2 < s < \delta - 1/2$ and $\mathcal{E} \subset \mathcal{N}$ ([10]). The operator H is said be of *generic type* if $\mathcal{N} = \{0\}$ and of *exceptional type* otherwise. When H is of *generic type*, we have rather satisfactory results (though there is much space for improvement in the condition on V) and it has been proved that $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ if $m \geq 3$ and, for all $1 < p < \infty$ if $m = 1$ and $m = 2$ under various smoothness and decay at infinity assumptions on V (see [4] for the best result when $m = 3$); but they are in general *not* bounded in $L^1(\mathbb{R}^1)$ or $L^\infty(\mathbb{R}^1)$ when $m = 1$ ([22]). When H is of exceptional type, it is long known that the same results hold when $m = 1$ (see [22, 3, 6]), however, we still poorly understand the problem when $m \geq 2$. It is proved that $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$ for $p \in (m/(m-2), m/2)$ when $m \geq 5$ and for $3/2 < p < 3$ when $m = 3$ ([7, 27], see also [11] for a partial result for $m = 4$) but nothing more to the best knowledge of the author. The purpose of this paper is to improve this situation by proving that $W_\pm$ are actually bounded in $L^p(\mathbb{R}^m)$ for a wider range of p's when $m = 3$ and $m \geq 5$. The cases $m = 2, 4$ and the end point problem are left for future investigation.

Throughout the paper, we assume that V is real measurable function which satisfies the following conditions, where $m_* = (m-1)/(m-2)$ and $C > 0$ and $\varepsilon > 0$ are constants:

$$\mathcal{F}(\langle x \rangle^{2\sigma} V) \in L^{m_*} \quad \text{for some } \sigma > 1/m_*. \tag{1.8}$$

$$|V(x)| \leq C\langle x \rangle^{-\gamma}, \quad \gamma = \begin{cases} m + 4 + \varepsilon, & \text{if } \ 3 \leq m \leq 7, \\ m + 3 + \varepsilon & \text{if } \ m \geq 8. \end{cases} \tag{1.9}$$

Note that the condition (1.8) requires certain smoothness for V.

Theorem 1.1. *Let $m \geq 3$. Suppose that V satisfy* (1.8) *and* (1.9) *and that H is of exceptional type. Then:*

(1) *For any $0 \leq k \leq 2$ and p such that* $\begin{cases} 1 < p < 3, & \text{if } m = 3, \\ 1 < p < m/2, & \text{if } m \geq 5 \end{cases}$

$$\|W_\pm u\|_{W^{k,p}} \leq C_p \|u\|_{W^{k,p}}, \qquad u \in W^{k,p}(\mathbb{R}^m) \cap L^2(\mathbb{R}^m) \tag{1.10}$$

(2) *If $m = 3$ and $p > 3$ or if $m \geq 5$ and $p > m/2$, then $W_\pm \notin \mathbf{B}(L^p(\mathbb{R}^m))$.*

We should emphasize that we have no results for p at the end points of the intervals and that the problem is completely open when $m = 2, 4$.

The rest of the paper is devoted to the proof of Theorem 1.1. We shall be concentrated in the proof of statement (1) because the second statement is contained in the result of Murata ([16], Theorem 1.2) which implies in particular that there exists a $u \in C_0^\infty(\mathbb{R}^m)$ such that

$$\lim_{t \to \infty} t^{m\left(\frac{1}{2} - \frac{1}{p}\right)} \|e^{-itH} P_{ac}(H) u\|_p = \infty \tag{1.11}$$

for any $p > 3$ if $m = 3$ and for any $p \in (m/2, \infty)$ if $m \geq 5$, which contradicts with (1.10) which would imply

$$\|e^{-itH} P_{ac}(H) u\|_p \leq C t^{m\left(\frac{1}{2} - \frac{1}{p}\right)} \|u\|_{p'}, \tag{1.12}$$

for the dual exponent p' of p by virtue of the well known L^p-L^q estimate for the free propagator e^{-itH_0} and the intertwing property (1.3). We also remark that statement (1) of the theorem follows if we prove

$$\|W_\pm u\|_{L^p} \leq C_p \|u\|_{L^p}, \quad u \in C_0^\infty(\mathbb{R}^m). \tag{1.13}$$

In fact the norms $\|(H_0 + c^2)u\|_{L^p}$ and $\|u\|_{W^{2,p}}$ are equivalent for any $1 < p < \infty$ and $c > 0$ via the Riesz transform and $\|(H + c^2)u\|_p$ and $\|(H_0 + c^2)u\|_p$ are also equivalent for sufficiently large $c > 0$ since we have $(H + V + c^2) = (1 + V(H_0 + c^2)^{-1})(H_0 + c^2)$ and $(1 + V(H_0 + c^2)^{-1})$ is an isomorphism of $L^p(\mathbb{R}^m)$. Thus, the intertwining property and (1.13) imply

$$\begin{aligned} \|W_\pm u\|_{W^{2,p}} &\leq C\|(H + i)W_\pm u\|_{L^p} = \|W_\pm (H_0 + i) u\|_{L^p} \\ &\leq C_p \|(H_0 + i)u\|_{L^p} \leq C\|u\|_{W^{2,p}}, \quad C_0^\infty(\mathbb{R}^m), \end{aligned} \tag{1.14}$$

and the estimate (1.10) for $k = 1$ then follows by interpolation. Thus, we shall be devoted in what follows to proving (1.13), the L^p-boundedness of $W_\pm$.

We use the following notation and conventions:

$$f \le_{|\cdot|} g \text{ means } |f| \le |g|.$$

For functions u and v, whenever $\overline{u(x)}v(x)$ is integrable we write

$$\langle u, v\rangle = \int_{\mathbb{R}^n} \overline{u(x)}v(x)dx.$$

We use this notation when v is in a certain function space and u is in its dual space. The rank 1 operator $\phi \mapsto \langle v, \phi\rangle u$ is interchangeably denoted by

$$|u\rangle\langle v| = u \otimes v.$$

For Banach spaces X and Y, $\mathbf{B}(X, Y)$ is the Banach space of bounded operators from X to Y and $\mathbf{B}(X) = \mathbf{B}(X, X)$. $\mathbf{B}_\infty(X, Y)$ and $\mathbf{B}_\infty(X)$ are spaces of compact operators. The identity operators in various Banach spaces are indistinguishably denoted by 1. For $1 \le p \le \infty$, $\|u\|_p = \|u\|_{L^p}$ is the norm of the Lebesgue spaces $L^p(\mathbb{R}^m)$. When $p = 2$, we often omit p and write $\|u\|$ for $\|u\|_2$. For $s \in \mathbb{R}$,

$$L^2_s = \langle x\rangle^{-s}L^2 = L^2(\mathbb{R}^m, \langle x\rangle^{2s}dx), \quad H^s(\mathbb{R}^m) = \mathcal{F}L^2_s(\mathbb{R}^m)$$

are the weighted L^2 spaces and Sobolev spaces. The space of rapidly decreasing functions is denoted by $\mathcal{S}(\mathbb{R}^m)$.

We denote the resolvents of H and H_0 respectively by

$$R(z) = (H - z)^{-1}, \quad R_0(z) = (H_0 - z)^{-1}.$$

We parameterize $z \in \mathbb{C} \setminus [0, \infty)$ as $z = \lambda^2$ by $\lambda \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ so that the boundaries $\{\lambda\colon \pm\lambda \in (0, \infty)\}$ are mapped onto the upper and lower edge of the positive half line $\{z \in \mathbb{C}\colon z > 0\}$. We define

$$G(\lambda) = R(\lambda^2), \quad G_0(\lambda) = R_0(\lambda^2), \quad \lambda \in \mathbb{C}^+.$$

These are $\mathbf{B}(\mathcal{H})$-valued meromorphic functions of $\lambda \in \mathbb{C}^+$ and the limiting absorption principle [15] (LAP for short) asserts that, when considered as $\mathbf{B}(\langle x\rangle^{-s}L^2, \langle x\rangle^{-t}L^2)$-valued functions, $s, t > \frac{1}{2}$ and $s + t > 2$, $G_0(\lambda)$ has the locally Hölder continuous extensions to its closure $\overline{\mathbb{C}}^+ = \{z : \Im z \ge 0\}$. The same is true also for $G(\lambda)$, but, if H is of exceptional type, it has singularities at $\lambda = 0$. In what follows $z^{\frac{1}{2}}$ is the branch of square root of z cut along the negative real axis such that $z^{\frac{1}{2}} > 0$ when $z > 0$. We assume $m \ge 3$ in what follows.

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2 Reduction to the low energy analysis

We study W_- only and denote it by W for simplicity. When $u \in \langle x \rangle^{-s} L^2$, $s > 1/2$, we may represent Wu as the limit

$$
\begin{aligned}
Wu &= u - \lim_{\varepsilon \downarrow 0, N \uparrow \infty} \frac{1}{\pi i} \int_{\varepsilon}^{N} G(\lambda) V (G_0(\lambda) - G_0(-\lambda)) u \lambda d\lambda && (2.1) \\
&= u - \frac{1}{\pi i} \int_0^\infty G(\lambda) V (G_0(\lambda) - G_0(-\lambda)) u \lambda d\lambda && (2.2)
\end{aligned}
$$

by using the boundary values of the resolvents ([15]). Here we understand the integral on the right of (2.1) as the Riemann integral of an $\langle x \rangle^t L^2$-valued continuous function, where $t > 1/2$ is such that $s + t > 2$. Then, the result of integral belongs to $L^2(\mathbb{R}^m)$, the limit exists in $L^2(\mathbb{R}^m)$ and the equation (2.1) is satisfied, which we symbolically write as (2.2).

We decompose W into the high and the low energy parts

$$
W = W_> + W_< \equiv W\Psi(H_0) + W\Phi(H_0), \tag{2.3}
$$

by using cut off functions $\Phi \in C_0^\infty(\mathbb{R})$ and $\Psi \in C^\infty(\mathbb{R})$ such that

$$
\Phi(\lambda^2) + \Psi(\lambda^2) \equiv 1, \quad \Phi(\lambda^2) = 1 \text{ near } \lambda = 0 \text{ and } \Phi(\lambda^2) = 0 \text{ for } |\lambda| > \lambda_0
$$

for a small constant $\lambda_0 > 0$ and we study $W_<$ and $W_>$ separately. We have proven in previous papers [27, 7] that, under the assumption of this paper, the high energy part $W_>$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 \le p \le \infty$ if $m \ge 3$ and we have nothing to add in this paper for $W_>$. Thus, in what follows, we shall be devoted to studying the low energy part $W_<$

$$
W_< = \Phi(H_0)u - \frac{1}{\pi i} \int_0^\infty G(\lambda) V (G_0(\lambda) - G_0(-\lambda)) \lambda \Phi(H_0) d\lambda. \tag{2.4}
$$

Since $\Phi(H_0)$ is evidently bounded in $L^p(\mathbb{R}^m)$ for all $1 \le p \le \infty$, we have only to study the integral part of (2.4) which we may write in the form

$$
Zu = -\frac{1}{\pi i} \int_0^\infty G_0(\lambda) V (1 + G_0(\lambda) V)^{-1} (G_0(\lambda) - G_0(-\lambda)) \lambda F(\lambda) u d\lambda \tag{2.5}
$$

by using Lemma 3.7 below. where $F(\lambda) = \Phi(\lambda^2)$. When H is of generic type, we have shown also in [27, 7] that Z is bounded in $L^p(\mathbb{R}^m)$ for all $1 \le p \le \infty$ under the condition of Theorem 1.1. Therefore, *we assume in the rest of the paper that H is of exceptional type.*

2.1 Low energy asymptotics. Further reduction

Since $\delta > 2$, the LAP (cf. Lemma 2.2 of [27]) implies that $G_0(\lambda)V$ is a locally Hölder continuous function of $\lambda \in \mathbb{R}$ with values in $\mathbf{B}_\infty(L^{-s})$ for any $1/2 < s < \delta - 1/2$ and, the absence of positive eigenvalues ([12]) implies that $1 + G_0(\lambda)V$ is invertible for $\lambda > 0$ (cf. [1]). It follows from the resolvent equation $G(\lambda) = G_0(\lambda) - G_0(\lambda)VG(\lambda)$ that

$$G(\lambda)V = G_0(\lambda)V(1 + G_0(\lambda)V)^{-1} \text{ for } \lambda \neq 0 \tag{2.6}$$

and, it also is $\mathbf{B}_\infty(L^{-s})$-valued locally Hölder continuous for $\lambda \in \mathbb{R} \setminus \{0\}$. However, if H is of exceptional type $\mathcal{N} = \mathrm{Ker}_{L^{-s}}(1 + G_0(0)V) \neq 0$, and $(1+G_0(\lambda)V)^{-1}$ has singularities at $\lambda = 0$. We determine their singularities by recalling some results from [27] and [7] and further reduce the problem to the study of Z_s which is obtained by inserting the singular part of $(1+G_0(\lambda)V)^{-1}$ into (2.5).

Since $G_0(0)V \in \mathbf{B}_\infty(L^2_{-s})$ with real integral kernel $C_m|x-y|^{2-m}V(y)$, $C_m > 0$, $\mathcal{N}$ is of finite dimensional and we may choose real valued functions as a basis. Functions $u \in \mathcal{N}$ satisfy the equation

$$-\Delta u + Vu = 0, \tag{2.7}$$

hence $u \in \langle x \rangle^{-s} H^2(\mathbb{R}^m)$ for any $s > 1/2$ and, moreover,

$$|u(x) \leq C\langle x \rangle^{2-m}, \text{ hence, in particular } \; Vu \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^m) \tag{2.8}$$

(see e.g. [2], corollary 2.6), however, it may not be an eigenfunction if $m \leq 4$.

2.1.1 Odd dimensional cases

The structure of singularities are different for different m. For odd dimensions $m \geq 3$ we have the following results (see, e.g. Theorem 2.12 of [27]). We state it separately for $m = 3$ and $m \geq 5$.

The case $m = 3$ If $m = 3$, $u \in \mathcal{N}$ satisfies

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{V(y)u(y)}{|x-y|} dy.$$

It follows that

$$u(x) = \frac{a(u)}{|x|} + O(|x|^{-2}) \text{ as } |x| \to \infty, \quad a(u) = \frac{1}{4\pi} \int_{\mathbb{R}^3} V(x)u(x)dx. \tag{2.9}$$

Thus, $\mathcal{E} = \{u \in \mathcal{N} \setminus \{0\} \colon a(u) = 0\}$ and, as $\mathcal{N} \ni u \mapsto a(u) \in \mathbb{C}$ is a continuous functional, $\dim \mathcal{N}/\mathcal{E} \le 1$. Any $\varphi \in \mathcal{N} \setminus \mathcal{E}$ is called *threshold resonance* of H. We say that H is of exceptional type of *the first kind* if $\mathcal{E} = \{0\}$, *the second* if $\mathcal{E} = \mathcal{N}$ and *the third kind* if $\{0\} \subsetneq \mathcal{E} \subsetneq \mathcal{N}$. The orthogonal projection in $\mathcal{H}$ onto the eigenspace $\mathcal{E}$ will be denoted by P_0. We let $D_0, D_1, \ldots$ be the integral operators defined by

$$D_j u(x) = \frac{1}{4\pi j!} \int_{\mathbb{R}^3} |x-y|^{j-1} u(y) dy, \quad j = 0, 1, \ldots .$$

so that we have a formal Taylor expansion

$$G_0(\lambda) u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i\lambda|x-y|}}{|x-y|} u(y) dy = \sum_{j=1}^{\infty} (i\lambda)^j D_j u.$$

If H is of exceptional type of the third kind, there a unique $\psi \in \mathcal{N}$ such that

$$-(V\psi, u) = 0, \quad \forall u \in \mathcal{E}, \quad -(V\psi, \psi) = 1 \text{ and } a(\psi) > 0.$$

We define

$$\varphi = \psi + P_0 V D_2 V \psi \in \mathcal{N}$$

and call it *the canonical resonance*. If H is of exceptional type of the first kind, then $\dim \mathcal{N} = 1$ and there is a unique $\varphi \in \mathcal{N}$ such that $-(V\varphi, \varphi) = 1$ and $a(\varphi) > 0$ and we call this the canonical resonance. We have the following result for $m = 3$ (see e.g. [27]). $\mathbf{B}_2(\mathcal{H})$ is the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$.

Theorem 2.1. *Let $m = 3$ and let V satisfy $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 3$. If H is of exceptional type of the third kind, φ is the canonical resonance and $a = 4\pi i |\langle V, \varphi\rangle|^{-2}$, Then, for any $s \in (3/2, \delta - 3/2)$,*

$$(I + G_0(\lambda)V)^{-1} = \frac{P_0 V}{\lambda^2} - \frac{P_0 V D_3 V P_0 V}{\lambda} - \frac{a}{\lambda} |\varphi\rangle\langle\varphi| V + I + L(\lambda). \quad (2.10)$$

$$\lambda \mapsto \langle x\rangle^{-s} L(\lambda) \langle x\rangle^{\delta - s} \in \mathbf{B}_2(\mathcal{H}) \text{ is of class } C^{\sigma - \frac{3}{2}} \text{ for any } \sigma < s. \quad (2.11)$$

If H is of exceptional type of the first or the second kind, (2.10) *and* (2.11) *hold with $P_0 = 0$ or $\varphi = 0$ respectively.*

The case $m \ge 5$ If $m \ge 5$, (2.8) implies there are no threshold resonances and $\mathcal{N} = \mathcal{E}$. We write P_0 for the orthogonal projection in $\mathcal{H}$ onto $\mathcal{E}$.

Theorem 2.2. *Let $m \geq 5$ be odd and $|V(x)| \leq C\langle x \rangle^{-\delta}$ for some $\delta > m+3$. Suppose H is of exceptional type. Then, there exists an $L(\lambda)$ which satisfies the property* (2.11) *for any $s \in (3/2, \delta - 3/2)$ with $\mathbf{B}_\infty(\mathcal{H})$ replacing $\mathbf{B}_2(\mathcal{H})$ such that, if $m = 5$,*

$$(I + G_0(\lambda)V)^{-1} = \frac{P_0 V}{\lambda^2} - \frac{c_0}{\lambda}|\varphi\rangle\langle\varphi|V + I + L(\lambda), \tag{2.12}$$

with $\varphi = P_0 V$, V being considered as a function, and $c_0 = i/(24\pi^2)$; and if $m \geq 7$,

$$(I + G_0(\lambda)V)^{-1} = \frac{P_0 V}{\lambda^2} + I + L(\lambda). \tag{2.13}$$

Decomposition of Z for odd dimensions. Write the right of (2.10), (2.12) and (2.13) as $S(\lambda) + I + L(\lambda)$ and insert this for $(I + G_0(\lambda)V)^{-1}$ in the right side of (2.5). This yields $Zu = Z_{r1}u + Z_{r2}u + Z_s u$ where

$$Z_{r1} = \frac{i}{\pi}\int_0^\infty G_0(\lambda)V(G_0(\lambda) - G_0(-\lambda))F(\lambda)\lambda d\lambda, \tag{2.14}$$

$$Z_{r2} = \frac{i}{\pi}\int_0^\infty G_0(\lambda)VL(\lambda)(G_0(\lambda) - G_0(-\lambda))F(\lambda)\lambda d\lambda, \tag{2.15}$$

$$Z_s = \frac{i}{\pi}\int_0^\infty G_0(\lambda)VS(\lambda)(G_0(\lambda) - G_0(-\lambda))F(\lambda)\lambda d\lambda. \tag{2.16}$$

We have shown that $Z_{r1}, Z_{r2} \in \mathbf{B}(L^p)$ for any $1 \leq p \leq \infty$ in [24] and in Sec. 3.1 of [27] respectively. Thus, when $m \geq 3$ is odd, we have only to investigate the operator Z_s.

2.2 Even dimensional case

In even dimensions, singular terms may contain logarithmic factors. The following theorem is Proposition 3.6 of [7]. We let $\dim \mathcal{E} = d$.

Theorem 2.3. *Let $m \geq 6$ be even. Suppose $|V(x)| \leq C\langle x \rangle^{-\delta}$ for $\delta > m+4$ if $m = 6$ and for $\delta > m+3$ if $m \geq 8$. Then, with an operator valued function $E(\lambda)$ which produces a bounded operators in $L^p(\mathbb{R}^m)$ for any $1 \leq p \leq \infty$ when inserted into* (2.5) *for $(1 + G_0(\lambda)V)^{-1}$, we have the following statements:*

(1) *If $m = 6$, then we have*

$$(1 + G_0(\lambda)V)^{-1} = \frac{P_0 V}{\lambda^2} + \sum_{j=0,1}\sum_{k=1,2} D_{jk}\lambda^j \log^k \lambda + E(\lambda), \tag{2.17}$$

where all D_{jk} are of rank at most $2d$ and VD_{jk} are of the form

$$VD_{jk} = \sum_{a,b=1}^{2d} \varphi_a \otimes \psi_b, \quad \varphi_a,\ \psi_b \in \langle x \rangle^{-\delta+3+\varepsilon} H^2(\mathbb{R}^6), \quad \forall \varepsilon > 0. \tag{2.18}$$

(2) *If $m \geq 8$, with a constant c_m and function $\varphi = P_0 V$ with V being considered as a function,*

$$(1 + G_0(\lambda)V)^{-1} = \frac{P_0 V}{\lambda^2} + c_m \varphi \otimes (V\varphi)\lambda^{m-6} \log \lambda + E(\lambda). \tag{2.19}$$

If $m \geq 12$, then $c_m \varphi \otimes (V\varphi)\lambda^{m-6} \log \lambda$ may be included in $E(\lambda)$.

Remark 2.4. In fact for $m = 6$, we have proven in [7] only that D_{jk} are of the form $D_{jk} = P_0 V D_{jk}^{(1)} P_0 V + D_{jk}^{(2)} P_0 V + P_0 V D_{jk}^{(3)}$ and, for any $\varepsilon > 0$,

$$D_{jk}^{(1)} \in \mathbf{B}(\mathcal{N}),\ D_{jk}^{(2)} \in \mathbf{B}(\mathcal{N}, \langle x \rangle^{-3-\varepsilon} L^2),\ D_{jk}^{(3)} \in \mathbf{B}(\langle x \rangle^{-\delta+3+\varepsilon} L^2, \mathcal{N}). \tag{2.20}$$

However, $1 + G_0(\lambda)V$ is a locally Hölder continuous function of $\lambda \in \mathbb{R}$ with values in $\mathbf{B}(\langle x \rangle^{-s} H^2(\mathbb{R}^6))$ and the proof of Proposition 3.6 goes through with $\langle x \rangle^{-\gamma} H^2(\mathbb{R}^6)$ replacing $\mathcal{H}_{-\gamma}$ everywhere, which implies that (2.20) holds with $H^2(\mathbb{R}^6)$ replacing L^2. This implies (2.18) by virtue of Riesz representation theorem.

By inserting (2.17) and (2.19) for $(I + G_0(\lambda)V)^{-1}$ in the right of (2.5), we have the decomposition $Zu = Z_r u + Z_{\log} u + Z_s u$ as in the case m is odd and

$$Z_r = \frac{i}{\pi} \int_0^\infty G_0(\lambda) V E(\lambda)(G_0(\lambda) - G_0(-\lambda)) F(\lambda) \lambda d\lambda$$

is bounded in $L^p(\mathbb{R}^m)$ for any $1 \leq p \leq \infty$ ([7]). Thus, we need study

$$Z_{\log} = \sum_{j,k} \frac{i}{\pi} \int_0^\infty G_0(\lambda) V \lambda^j (\log \lambda)^k D_{jk} (G_0(\lambda) - G_0(-\lambda)) F(\lambda) \lambda d\lambda, \tag{2.21}$$

$$Z_s = \frac{i}{\pi} \int_0^\infty G_0(\lambda) V P_0 V (G_0(\lambda) - G_0(-\lambda)) F(\lambda) \lambda^{-1} d\lambda \tag{2.22}$$

in what follows for even $m \geq 6$.

3 Preliminaries

We record several results which we need in what follows and which are mostly well known.

3.1 Results from harmonic analysis.

The following is the Muckenhaupt weighted inequality (cf. [8], Chapter 9).

Lemma 3.1. *The weight $|r|^a$ is an A_p weight on $\mathbb{R}$ if and only if $-1 < a < p-1$. The Hilbert transform $\tilde{\mathcal{H}}$ and the Hardy-Littlewood maximal operator $\mathcal{M}$ are bounded in $L^p(\mathbb{R}, w(r)dr)$ for A_p weights $w(r)$.*

We shall repeatedly use the fact that

$$|r|^{m-1-p(m-1)}, \quad |r|^{m-1-2p} \quad \text{and} \quad |r|^{2-p}$$

are A_p weights on $\mathbb{R}^1$ respectively for $1 < p < m/(m-1)$ ($m \geq 2$), for $m/3 < p < m/2$ ($m \geq 3$) and for $3/2 < p < 3$.

For a function $F(x)$ on $\mathbb{R}^m$, $G(|x|) \in L^1(\mathbb{R}^m)$ is said to be a radial decreasing integrable majorant (RDIM) of F if $G(r) > 0$ is a decreasing function of $r > 0$, and $|F(x)| \leq G(|x|)$ for a.e. $x \in \mathbb{R}^m$. The following lemma may be found e.g. on page 57 of [21].

Lemma 3.2. (1) *A rapidly decreasing function $F \in \mathcal{S}(\mathbb{R}^m)$ has a RDIM.*

(1) *If F has a RDIM. then there is a constant $C > 0$ such that*

$$|(F * u)(t)| \leq C(\mathcal{M}u)(t), \quad t \in \mathbb{R}. \tag{3.1}$$

We define the operator $\mathcal{H}$ on $\mathbb{R}$ by

$$\mathcal{H}u(\rho) = \frac{(1 + \tilde{\mathcal{H}})u(\rho)}{2} = \frac{1}{2\pi}\int_0^\infty e^{ir\rho}\hat{u}(r)dr \tag{3.2}$$

where $\tilde{\mathcal{H}}$ is the Hilbert transform.

Lemma 3.3. *For u and $F \in L^1(\mathbb{R})$ such that $\hat{u}$, $\hat{F} \in L^1(\mathbb{R})$ we have*

$$\frac{1}{2\pi}\int_0^\infty e^{i\lambda\rho}F(\lambda)\hat{u}(\lambda)d\lambda = (\mathcal{F}^*F * \mathcal{H}u)(\rho). \tag{3.3}$$

Proof. Let $\Theta(\lambda) = \begin{cases} 1, & \text{for } \lambda > 0 \\ 0, & \text{for } \lambda \leq 0 \end{cases}$. Then, the left side equals

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\rho}F(\lambda)\Theta(\lambda)\hat{u}(\lambda)d\lambda &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{i\lambda(\rho-\xi)}\mathcal{F}^*F(\xi)d\xi\right)\Theta(\lambda)\hat{u}(\lambda)d\lambda \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}^*F(\xi)\mathcal{F}\{\Theta(\lambda)\hat{u}(\lambda)\}(\rho-\xi)d\xi = (\mathcal{F}^*F * \mathcal{H}u)(\rho)\end{aligned}$$

as desired. ∎

3.2 Resolvent kernel

The resolvent $G_0(\lambda)$ for $\Im\lambda \geq 0$ is a convolution operator and the convolution kernel is given for $m \geq 2$ by the Whittaker formula (cf. [23])

$$G_0(\lambda, x) = \frac{e^{i\lambda|x|}}{2(2\pi)^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)|x|^{m-2}} \int_0^\infty e^{-t} t^{\frac{m-3}{2}} \left(\frac{t}{2} - i\lambda|x|\right)^{\frac{m-3}{2}} dt. \quad (3.4)$$

When $m \geq 3$ is odd, $\frac{m-3}{2}$ is an integer and we have the following expression.

Lemma 3.4. *Let $m \geq 3$ be odd. Then, $G_0(\lambda, x)$ is an exponential polynomial like function:*

$$G_0(\lambda, x) = \sum_{j=0}^{(m-3)/2} C_j \frac{(\lambda|x|)^j e^{i\lambda|x|}}{|x|^{m-2}} \quad \textit{with } C_j = \frac{(-i)^j (m-3-j)!}{2^{m-1-j}\pi^{\frac{m-1}{2}} j! (\frac{m-3}{2} - j)!}. \quad (3.5)$$

and the coefficients C_0 and C_1 satisfy

$$iC_0 + C_1 = 0, \quad \textit{when } m \geq 5. \quad (3.6)$$

If m is even, then $\frac{m-3}{2}$ is a half integer and derivatives of $G_0(\lambda, x)$ become singular at $\lambda = 0$. This makes the analysis for even dimensional cases considerably more complex than for odd dimensional ones. Nevertheless, the expression given below of $G_0(\lambda, x)$ for even dimensions $m \geq 4$ as a superposition of exponential polynomial like functions allows some arguments for even dimensions to go in parallel to the ones used for odd dimensional cases. We set

$$\nu = \frac{m-2}{2}$$

and define superposing operators $T_j^{(a)}$ over parameter $a > 0$ for $j = 0, \ldots, \nu$ by

$$T_j^{(a)}[f(x, a)] = C_{m,j} \int_{\mathbb{R}_+} (1+a)^{-(2\nu - j + \frac{1}{2})} f(x, a) \frac{da}{\sqrt{a}}, \quad (3.7)$$

$$C_{m,j} = (-2i)^j \frac{\Gamma\left(2\nu - j + \frac{1}{2}\right)}{2^{m-1}\pi^{\frac{m}{2}}\Gamma\left(\frac{m-1}{2}\right)} \binom{\nu}{j}. \quad (3.8)$$

Notice that $2\nu - j \geq 1$ for $m \geq 4$ and the integral (3.7) converges absolutely if f is bounded with respect to a.

Lemma 3.5. *If $m \geq 4$ is even, then we have*

$$G_0(\lambda, x) = \sum_{j=0}^{\nu} T_j^{(a)} \left[e^{i\lambda|x|(1+2a)} \frac{(\lambda|x|)^j}{|x|^{m-2}} \right]. \quad (3.9)$$

Proof. Write the Whittaker formula (3.4) for $G_0(\lambda, x)$ in the form

$$G_0(\lambda, x) = \frac{e^{i\lambda|x|}}{C_m|x|^{m-2}} \int_0^\infty e^{-t} t^{\frac{m-3}{2}} (t - 2i\lambda|x|)^{\frac{m-3}{2}} \, dt, \tag{3.10}$$

where $C_m = 2^{m-1}\pi^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)$. In the integrand of (3.10) we write

$$(t - 2i\lambda|x|)^{\frac{m-3}{2}} = (t - 2i\lambda|x|)^{\nu}(t - 2i\lambda|x|)^{-\frac{1}{2}},$$

expand $(t - 2i\lambda|x|)^{\nu}$ via the binomial formula and use the identity

$$z^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-az} \, a^{-\frac{1}{2}} \, da, \quad \Re\, z > 0 \tag{3.11}$$

for $(t - 2i\lambda|x|)^{-\frac{1}{2}}$. Thus, the right hand side of (3.10) becomes

$$\sum_{j=0}^{\nu} \frac{(-2i)^j}{\sqrt{\pi}C_m} \binom{\nu}{j} \iint_{\mathbb{R}^2_+} e^{-(1+a)t} t^{2\nu-j} \left(e^{i\lambda|x|(1+2a)} \frac{(\lambda|x|)^j}{|x|^{m-2}} \right) \frac{dt}{\sqrt{t}} \frac{da}{\sqrt{a}}.$$

The integral converges absolutely if $m \geq 4$ and this is nothing but (3.9). $\square$

Lemma 3.6. *For $m \geq 4$, $G_0(\lambda, x)$ may also be written in the form*

$$G_0(\lambda, x) = \sum_{j=0}^{\nu} \frac{(\lambda|x|)^j e^{i|x|\lambda}}{|x|^{m-2}} G_{j,m}(\lambda|x|), \tag{3.12}$$

where $G_{j,m} \in C^\infty(0, \infty) \cap C^{m-j-3}([0, \infty))$ and it satisfies for $\rho > 1$ that

$$|G_{j,m}^{(l)}(\rho)| \leq C_l \rho^{-l-\frac{1}{2}}, \quad l = 0, 1, \ldots. \tag{3.13}$$

Proof. By changing the contour of integration, we have

$$\begin{aligned} G_{j,m}(\rho) &= C_{jm} \int_0^\infty e^{-2a\rho}(1 + ia)^{-(2\nu-j+\frac{1}{2})} \frac{da}{\sqrt{a}} \\ &= \frac{C_{jm}}{\sqrt{\rho}} \int_0^\infty e^{-2a} \left(1 + \frac{ia}{\rho}\right)^{-(2\nu-j+\frac{1}{2})} \frac{da}{\sqrt{a}}. \end{aligned}$$

The lemma is obvious from these expressions. $\square$

3.3 Spectral measure of $-\Delta$

.

Let $E_0(d\mu)$ be the spectral measure for $-\Delta$. Then, with $\mu = \lambda^2$,

$$E_0(d\mu) = \frac{1}{2\pi i}(R_0(\mu + i0) - R_0(\mu - i0))d\mu = \frac{1}{i\pi}(G_0(\lambda) - G_0(-\lambda))\lambda d\lambda.$$

We set $F(\lambda) = \Phi(\lambda^2)$ for $\lambda \in \mathbb{R}$. Then, $F \in C_0^\infty(\mathbb{R})$ and $F(|D|) = \Phi(H_0)$.

Lemma 3.7. *Let $m \geq 3$. Then, for $u, v \in \mathcal{S}(\mathbb{R}^m)$, both sides of the following equation can be continuously extended to $\lambda = 0$ and*

$$\lambda^{-1}\langle v, (G_0(\lambda) - G_0(-\lambda))u\rangle = \langle |D|^{-1}v, (G_0(\lambda) - G_0(-\lambda))u\rangle, \quad \lambda \geq 0. \tag{3.14}$$

For continuous function f of $\lambda \in \mathbb{R}$ with compact support, we have for $\lambda \geq 0$,

$$\langle v, G_0(\lambda)u - G_0(-\lambda))f(\lambda)u\rangle = \langle v, (G_0(\lambda)u - G_0(-\lambda))f(|D|)u\rangle \tag{3.15}$$

Proof. Let $\lambda > 0$. We have for any $u, v \in \mathcal{S}(\mathbb{R}^m)$ that

$$\langle v, (G_0(\lambda) - G_0(-\lambda))u\rangle = \frac{\lambda^{m-2}i}{2(2\pi)^{m-1}}\int_\Sigma \overline{\hat{v}(\lambda\omega)}\hat{u}(\lambda\omega)d\omega \tag{3.16}$$

It follows, since $\widehat{|D|^{-1}v}(\lambda\omega) = \lambda^{-1}\hat{v}(\lambda\omega)$, that

$$\langle |D|^{-1}v, (G_0(\lambda) - G_0(-\lambda))u\rangle = \frac{\lambda^{m-3}i}{2(2\pi)^{m-1}}\int_\Sigma \overline{\hat{v}(\lambda\omega)}\hat{u}(\lambda\omega)d\omega. \tag{3.17}$$

The right hand side obviously extends to $u, v \in L^1(\mathbb{R}^m)$ to produce continuous functions of $\lambda \geq 0$ when $m \geq 3$ and (3.14) follows by comparing (3.16) and (3.17). If f is continuous and bounded, then $\mathcal{F}f(|D|)u = f(|\xi|)\hat{u}(\xi)$ and (3.15) likewise follows. □

We define the spherical average of a function f on $\mathbb{R}^m$ by

$$M(r, f) = \frac{1}{|\Sigma|}\int_\Sigma f(r\omega)d\omega, \quad \text{for all } r \in \mathbb{R}. \tag{3.18}$$

Here $\Sigma = \mathbb{S}^{m-1}$ is the unit sphere and $|\Sigma|$ is its area. Hölder's inequality implies

$$\left(\int_0^\infty |M(r)|^p r^{m-1}dr\right)^{1/p} \leq \|f\|_p. \tag{3.19}$$

$M(r, f)$ is an even function of $r \in \mathbb{R}$. We then often use the following formula. For an even function $M(r)$ of $r \in \mathbb{R}$, define $\tilde{M}(\rho)$ by

$$\tilde{M}(\rho) = \int_\rho^\infty rM(r)dr\left(= -\int_{-\infty}^\rho rM(r)dr\right). \tag{3.20}$$

Lemma 3.8. *Suppose $M(r) = M(-r)$ and $\langle r\rangle^2 M(r)$ is integrable. Then,*

$$\int_{\mathbb{R}} e^{-ir\lambda} rM(r)dr = \frac{\lambda}{i}\int_{\mathbb{R}} e^{-ir\lambda}\tilde{M}(r)dr, \quad \int_{\mathbb{R}} \tilde{M}(r)dr = \int_{\mathbb{R}} r^2 M(r)dr. \quad (3.21)$$

Proof. Since $rM(r) = -\tilde{M}(r)'$, integration by parts gives the first equation. We differentiate both sides of the first and set $\lambda = 0$. The second follows. □

We denote $\check{u}(x) = u(-x)$, $x \in \mathbb{R}^m$. (The sign $\check{}$ will be reserved for this purpose and will not be used to denote the conjugate Fourier transform.)

Representation formula for odd dimensions.

Lemma 3.9. *Let $m \geq 3$ be odd. Let $\psi \in L^1(\mathbb{R}^m)$ and $u \in \mathcal{S}(\mathbb{R}^m)$. Let $c_j = |\Sigma|C_j$ where C_j are the constants in* (3.5)*. Then, for $\lambda > 0$ we have*

$$\langle \psi \,|\, (G_0(\lambda) - G_0(-\lambda))u\rangle = \sum_{j=0}^{(m-3)/2} c_j(-1)^{j+1}\lambda^j \int_{\mathbb{R}} e^{-i\lambda r} r^{1+j} M(r, \overline{\psi} * \check{u})dr. \quad (3.22)$$

Remark 3.10. Combining (3.16) with (3.22) we have the identity:

$$\frac{\pi i \lambda^{m-2}}{(2\pi)^m}\int_{\Sigma} \overline{\hat{\psi}(\lambda\omega)}\hat{u}(\lambda\omega)d\omega = \sum_{j=0}^{(m-3)/2} c_j(-1)^{j+1}\lambda^j \int_{\mathbb{R}} e^{-i\lambda r} r^{1+j} M(r, \overline{\psi} * \check{u})dr.$$

This is particularly simple for $m = 3$:

$$\int_{\Sigma} \overline{\hat{\psi}(\lambda\omega)}\hat{u}(\lambda\omega)d\omega = \frac{8\pi^2 i}{\lambda}\int_{\mathbb{R}} e^{-i\lambda r} rM(r, \overline{\psi} * \check{u})dr \quad (3.23)$$

Proof. We compute $\langle \psi \,|\, G_0(\lambda)u\rangle$ by inserting the integral kernel (3.5) of $G_0(\lambda)$. After changing the order of integration, we rewrite the integral by using polar coordinates. Then,

$$\begin{aligned}
\langle \psi \,|\, G_0(\lambda)u\rangle &= \sum_{j=0}^{(m-3)/2} C_j \int_{\mathbb{R}^m} \overline{\psi(x)} \left(\int_{\mathbb{R}^m} \frac{\lambda^j e^{i\lambda|y|}u(x-y)}{|y|^{m-2-j}}dy\right)dx \\
&= \sum_{j=0}^{(m-3)/2} C_j \int_{\mathbb{R}^m} \frac{\lambda^j e^{i\lambda|y|}(\overline{\psi} * \check{u})(y)}{|y|^{m-2-j}}dy \\
&= \sum_{j=0}^{(m-3)/2} c_j \int_0^\infty \lambda^j e^{i\lambda r} r^{1+j} M(r, \overline{\psi} * \check{u})dr, \ c_j = C_j|\Sigma|.
\end{aligned}$$

Since $M(r)$ is an even function, change of variable r to $-r$ yields

$$-\langle\psi\,|\,G_0(-\lambda)u\rangle = \sum_{j=0}^{(m-3)/2} c_j \int_{-\infty}^{0} \lambda^j e^{i\lambda r} r^{1+j} M(r, \overline{\psi} * \check{u})dr.$$

Add both sides of last two equations and change the variable r to $-r$. This produces (3.22). $\square$

Spectral measure in even dimensions. If m is even, we have the analogue of (3.22). For a function $A(r)$ on $\mathbb{R}$ and $a > 0$, define

$$A^a(r) = A(r/(1+2a))$$

and write $M_{\overline{\psi}*\check{u}}(r)$ for $M(r, \overline{\psi} * \check{u})$ for shorting the formula .

Lemma 3.11. *Let $m \geq 2$. Let $\psi \in L^1(\mathbb{R}^m)$ and $u \in \mathcal{S}(\mathbb{R}^m)$. Then*

$$\langle\psi, (G_0(\lambda) - G_0(-\lambda))u\rangle = \sum_{j=0}^{\nu} (-1)^{j+1} |\Sigma| T_j^{(a)} \left[\frac{\lambda^j \mathcal{F}(r^{j+1} M^a_{\overline{\psi}*\check{u}})(\lambda)}{(1+2a)^{j+2}}\right] \tag{3.24}$$

$$= \sum_{j=0}^{\nu} (-1)^{j+1} |\Sigma| T_j^{(a)} \left[\lambda^j \mathcal{F}(r^{j+1} M_{\overline{\psi}*\check{u}})((1+2a)\lambda)\right]. \tag{3.25}$$

The term with $j = 0$ in the right of (3.24) *may also be written as*

$$i|\Sigma| T_0^{(a)} \left[\frac{\lambda(\mathcal{F}\widetilde{M^a_{\overline{\psi}*\check{u}}})(\lambda)}{(1+2a)^2}\right]. \tag{3.26}$$

Proof. Define $A_j(\lambda, r, a) = e^{i\lambda r(1+2a)}(\lambda r)^j r^{-(m-2)}$ and operator $A_j(\lambda, a)$ by

$$A_j(\lambda, a)u(x) = \int_{\mathbb{R}^m} A_j(\lambda, |y|, a) u(x-y)dy, \quad j = 0, \ldots, \nu.$$

Then, (3.9) and change of the order of integration imply

$$\langle\psi, (G_0(\lambda) - G_0(-\lambda))u\rangle = \sum_{j=0}^{\nu} T_j^{(a)} \left[\langle\psi, (A_j(\lambda, a) - A_j(-\lambda, a))u\rangle\right]. \tag{3.27}$$

We have, as in odd dimensions, that for $u \in \mathcal{S}(\mathbb{R}^m)$ and $\psi \in L^1(\mathbb{R}^m)$

$$\langle\psi, A_j(\lambda, a)u\rangle = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \overline{\psi(x)} A_j(\lambda, |y|, a) u(x-y)dy\right) dx$$

$$= \int_{\mathbb{R}^m} A_j(\lambda, |y|, a)(\overline{\psi} * \check{u})(y)dy = |\Sigma| \int_0^\infty e^{i(1+2a)\lambda r}(\lambda r)^j r M_{\overline{\psi}*\check{u}}(r)dr.$$

Replacing λ to $-\lambda$ and changing the variable r to $-r$, we have

$$-\langle \psi, A_j(-\lambda, a)u\rangle = |\Sigma| \int_{-\infty}^0 e^{i(1+2a)\lambda r}(\lambda r)^j r M_{\overline{\psi}*\check{u}}(r)dr,$$

where we used that $M_{\overline{\psi}*\check{u}}(-r) = M_{\overline{\psi}*\check{u}}(r)$. Adding these two yields

$$\langle \psi, (A_j(\lambda, a) - A_j(-\lambda, a))u\rangle = |\Sigma| \int_{\mathbb{R}} e^{i(1+2a)\lambda r}(\lambda r)^j r M_{\overline{\psi}*\check{u}}(r)dr. \qquad (3.28)$$

Changing r to $-r$ in the right of (3.28) and plugging the result with (3.27), we obtain (3.25). If we change the variable r to $-r/(1+2a)$, (3.28) becomes

$$\frac{(-1)^{j+1}|\Sigma|}{(1+2a)^{j+2}} \int_{\mathbb{R}} e^{-i\lambda r}\lambda^j r^{j+1} M^{a}_{\overline{\psi}*\check{u}}(r)dr = \frac{(-1)^{j+1}|\Sigma|\lambda^j}{(1+2a)^{j+2}} \mathcal{F}(r^{j+1}M^{a}_{\overline{\psi}*\check{u}})(\lambda).$$

If we substitute this for $\langle \psi, (A_j(\lambda, a) - A_j(-\lambda, a))u\rangle$ in (3.27), (3.24) follows. If we use the second equation of (3.21) the right of the last equation for $j = 0$ becomes

$$\frac{i\lambda(\mathcal{F}\widetilde{M^{a}_{\overline{\psi}*\check{u}}})(\lambda)}{(1+2a)^2}|\Sigma|$$

and (3.26) follows. □

3.4 Proof of Theorem 1.1 for odd $m \geq 3$.

We substitute (2.10), (2.12) and (2.13) for $S(\lambda)$ in the equation (2.16) respectively for $m = 3, 5$ and $m \geq 7$. We write $\{\phi_1, \ldots, \phi_d\}$ for the orthonormal basis of the 0 eigenspace $\mathcal{E}$ of H and define

$$Z_{s1}u = \sum_{j=1}^{d} \frac{i}{\pi} \int_0^\infty G_0(\lambda)|V\phi_l\rangle\langle V\phi_l|(G_0(\lambda) - G_0(-\lambda))F(\lambda)\lambda^{-1}ud\lambda. \qquad (3.29)$$

Then, we have $Z_s = Z_{s1}$ if $m \geq 7$ and, if $m = 3$ and $m = 5$, we have less singular extra term Z_{s0} given below and $Z_s = Z_{s0} + Z_{s1}$: For $m = 5$, with $\varphi = P_0 V$

$$Z_{s0}u = \int_0^\infty G_0(\lambda)|\, V\varphi\rangle\langle V\varphi\,|\,(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda$$

and for $m = 3$, with ϕ_0 being the canonical resonance

$$Z_{s0}u = \sum_{l,n=0}^{d} a_{ln} \int_0^\infty G_0(\lambda)|\, V\phi_l\rangle\langle V\phi_n\,|\,(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda$$

where $a_{ln},\ 0 \le l, n \le d$ are constants such that $a_{0l} = a_{l0} = 0$ for $1 \le l \le d$. Thus, for proving Theorem 1.1 for odd $m \ge 3$, it suffices to prove the following proposition.

Proposition 3.12. *Let $1 < p < 3$ if $m = 3$ and $1 < p < m/2$ if $m \ge 5$.*

(1) *For any $\phi, \psi \in \mathcal{N}$, the operator $\tilde{Z}_{s0}(\phi,\psi)$ defined by*

$$\tilde{Z}_{s0}(\phi,\psi)u = \frac{i}{\pi}\int_0^\infty G_0(\lambda)V\phi\rangle\langle V\psi|(G_0(\lambda) - G_0(-\lambda))F(\lambda)ud\lambda \quad (3.30)$$

is bounded in $L^p(\mathbb{R}^m)$ if $m = 3, 5$.

(2) *For any $\phi, \psi \in \mathcal{E}$, the operator $\tilde{Z}_{s1}(\phi,\psi)$ defined by*

$$\tilde{Z}_{s1}(\phi,\psi)u = \frac{i}{\pi}\int_0^\infty G_0(\lambda)V\phi\rangle\langle V\psi|(G_0(\lambda)-G_0(-\lambda))F(\lambda)u\frac{d\lambda}{\lambda}. \quad (3.31)$$

is bounded in $L^p(\mathbb{R}^m)$ for all odd $m \ge 3$.

Proof. Write $\tilde{Z}_{s\ell}$ for $\tilde{Z}_{s\ell}(\phi,\psi)$, $\ell = 0, 1$ for simplicity. We replace $G_0(\lambda)$ by (3.5) and use (3.22) for $\langle V\psi, (G_0(\lambda) - G_0(-\lambda))u\rangle$. Writing $M(r)$ for $M(r, V\psi * u)$, we obtain

$$\tilde{Z}_{s\ell}u = \frac{i}{\pi}\sum_{j,k=0}^{\frac{m-3}{2}} (-1)^{j+1} C_k c_j Z_{s\ell}^{(j,k)}u, \quad (3.32)$$

where $Z_{s\ell}^{(j,k)}$, $0 \le j, k \le \frac{m-3}{2}$, are given by

$$Z_{s\ell}^{(j,k)}u(x) = \int_{\mathbb{R}^m} \frac{V\phi(y)}{|x-y|^{m-2-k}} K_\ell^{(j,k)}(|x-y|)dy, \quad (3.33)$$

$$K_\ell^{(j,k)}(\rho) = \frac{1}{2\pi}\int_0^\infty e^{i\lambda\rho}\lambda^{j+k-\ell}\left(\int_{\mathbb{R}} e^{-i\lambda r} r^{j+1} M(r)dr\right) F(\lambda)d\lambda. \quad (3.34)$$

By using Young's inequality and polar coordinates, we have from (3.33) that

$$\|Z_{s\ell}^{(j,k)}u\|_p \le C\|V\phi\|_1 \left(\int_0^\infty \frac{|K_\ell^{(j,k)}(\rho)|^p}{\rho^{(m-2-k)p}}\rho^{m-1}d\rho\right)^{1/p}. \quad (3.35)$$

This will often be the starting point of estimates though some modification will be in some cases. When $\ell = 0$, we have by virtue of Lemmas 3.3 and 3.2 that

$$K_0^{(j,k)}(\rho) = \{(\mathcal{F}^*(\lambda^{j+k}F) * \mathcal{H}(r^{j+1}M(r))\}(\rho) \le_{|\cdot|} C\mathcal{M}(r^{j+1}M)(\rho) \tag{3.36}$$

for all $0 \le j, k \le \frac{m-3}{2}$. The proposition is proved by the series of lemmas given below. □

Lemma 3.13. *Let $m = 3$ and $1 < p < 3$. Then, $\|Z_{s0}u\|_p \le C\|u\|_p$ for a constant C independent of $u \in C_0^\infty(\mathbb{R}^3)$.*

Proof. If $m = 3$, we have $j = k = 0$ only and we omit the suffix (j,k) from $Z_{s0}^{(j,k)}u$ and $K_0^{(j,k)}$. Let $3/2 < p < 3$ first. Then, ρ^{2-p} is an A_p weight on $\mathbb{R}$ and, applying Lemma 3.1 to (3.36), we obtain

$$\left(\int_0^\infty |K_0(\rho)|^p \rho^{2-p} d\rho\right)^{1/p} \le C\left(\int_0^\infty M(r)^p r^2 dr\right)^{1/p} \tag{3.37}$$

$$\le C\|V\psi * u\|_p \le C\|V\psi\|_1\|u\|_p. \tag{3.38}$$

The lemma for $3/2 < p < 3$ follows from this and (3.35). For $1 < p < 3/2$, we apply integration by parts and apply Lemmas 3.3 and 3.2 to the resulting equation and obtain

$$\begin{aligned} K_0(\rho) &= \frac{i}{2\pi\rho}\int_0^\infty e^{i\lambda\rho}\left(F(\lambda)\int_{\mathbb{R}} e^{-ir\lambda} rM(r)dr\right)' d\lambda \\ &\le_{|\cdot|} C\rho^{-1}(\mathcal{M}(r^2M)(\rho) + \mathcal{M}(rM)(\rho)). \end{aligned} \tag{3.39}$$

If $1 < p < 3/2$, then, ρ^{2-2p} is an A_p weight on $\mathbb{R}$ and (3.39) and Lemma 3.1 imply

$$\begin{aligned} &\left(\int_0^\infty |K_0(\rho)|^p \rho^{2-p} d\rho\right)^{1/p} \le \left(\int_0^\infty |\mathcal{M}(r^2M)(\rho) + \mathcal{M}(rM)(\rho)|^p \rho^{2-2p} d\rho\right)^{1/p} \\ &\le C\left(\int_0^\infty |M(r)|^p r^2 dr\right)^{1/p} + C\left(\int_0^\infty |M(r)|^p r^{2-p} dr\right)^{1/p}. \end{aligned} \tag{3.40}$$

We estimate the first on the right of (3.40) as in (3.38) and the second by

$$\begin{aligned} &\left(\int_0^\infty |M(r)|^p r^2 dr\right)^{1/p} + \left(\int_0^1 |M(r)|^p r^{2-p} dr\right)^{1/p} \\ &\le C(\|V\psi * u\|_p + \|V\psi * u\|_\infty) \le C(\|V\psi\|_1 + \|V\psi\|_{p'})\|u\|_p, \end{aligned} \tag{3.41}$$

where $p' = p/(p-1)$ is the dual exponent of p and we used obvious estimate

$$\sup |M(r)| \le \|V\psi * u\|_\infty. \tag{3.42}$$

Combining (3.41) with (3.35) we have the lemma for $1 < p < 3/2$ as well. Then, the interpolation theorem ([5]) completes the proof of the lemma. $\square$

Lemma 3.14. *Let $m = 5$ and $1 < p < 5/2$. Then, $\|\tilde{Z}_{s0}u\|_p \le C\|u\|_p$ for a constant C independent of $u \in C_0^\infty(\mathbb{R}^5)$.*

Proof. By virtue of the interpolation theorem, it suffices to prove the lemma for $1 < p < 5/4$ and for $2 \le p < 5/2$. Though $m = 5$ in this proof, we will also write m for the space dimension 5 as this is sometimes conveninet. We integrate by parts $k+1$-times in the right of (3.34) for $\ell = 0$. Since all derivatives up to the order k of $\lambda^{j+k}\left(\int_{\mathbb{R}} e^{-i\lambda r} r^{j+1} M(r)dr\right) F(\lambda)$ vanish at $\lambda = 0$, no boundary terms appear and after estimating the resulting equation by using Lemmas 3.3 and 3.2, we obtain in addition to (3.36) that

$$\begin{aligned}
K_0^{(j,k)}(\rho) &= \frac{i^{k+1}}{2\pi\rho^{k+1}} \int_0^\infty e^{i\lambda\rho} \left(\frac{\partial}{\partial\lambda}\right)^{k+1} \left(\lambda^{j+k} F(\lambda) \int_{\mathbb{R}} e^{-i\lambda r} r^{j+1} M(r)dr\right) d\lambda \\
&\le_{|\cdot|} \frac{C}{\rho^{k+1}} \sum_{l=0}^{k+1} \mathcal{MH}(r^{j+l+1}M(r))(\rho).
\end{aligned} \tag{3.43}$$

Let $1 < p < m/(m-1)$. Then, $|r|^{-(p-1)(m-1)}$ is an A_p weight on $\mathbb{R}$ and (3.43) and Lemma 3.1 imply that the integral of on the right of (3.35) is bounded by a constant times

$$\begin{aligned}
\sum_{l=0}^{k+1} \left(\int_0^\infty \frac{|\mathcal{MH}(r^{j+l+1}M)(\rho)|^p}{\rho^{(p-1)(m-1)}} d\rho\right)^{1/p} &\le C \sum_{l=0}^{k+1} \left(\int_0^\infty \frac{|r^{j+l+1}M(r)|^p}{r^{(p-1)(m-1)}} dr\right)^{1/p} \\
\le C\left(\int_{\mathbb{R}} \frac{|(|r| + |r|^{m-1})M(r)|^p}{r^{(p-1)(m-1)}} dr\right)^{1/p} &\le C(\|V\psi\|_1 + \|V\psi\|_{p'})\|u\|_p.
\end{aligned} \tag{3.44}$$

Here the last estimate is obtained as in (3.41) by using (3.42). This proves the lemma for $1 < p < 5/4$.

If $2 \le p < m/2$, then $|r|^{m-1-2p}$ is an A_p weight on $\mathbb{R}$ and $m-2-k = 2$ if $m = 5$ and $k = 1$. Hence, estimating the right of (3.35) by using (3.36) and Lemma 3.1 as previously, we obtain

$$\|Z_0^{(j,1)}u\|_p \le C\|V\phi\|_1 \left(\int_0^\infty |\mathcal{MH}(r^{j+1}M(r))|^p \rho^{m-1-2p} d\rho\right)^{1/p}$$

$$\leq C\|V\phi\|_1 \left(\int_0^\infty |M(r))|^p \rho^{m-1-(1-j)p} d\rho\right)^{1/p} \leq C\|u\|_p. \quad (3.45)$$

When $k=0$, we split the integral in (3.33) and estimate

$$Z_{s0}^{(j,0)} u(x) \leq_{|\cdot|} \int_{|x-y|\leq 1} \frac{|V\phi(y)|}{|x-y|^3} |K_0^{(j,0)}(|x-y|)| dy + \int_{|x-y|\geq 1} \frac{|V\phi(y)|}{|x-y|^2} |K_0^{(j,0)}(|x-y|)| dy = I_1(x) + I_2(x).$$

We have by using Young's inequality that

$$\|I_2\|_p \leq \|V\phi\|_1 \left\| \frac{K_0^{(j,0)}(|x|)}{|x|^2} \right\|_p \quad (3.46)$$

and, applying Lemma 3.1 to (3.36) again,

$$\left\| \frac{K_0^{(j,0)}(|x|)}{|x|^2} \right\|_p = C \left(\int_0^\infty |K_0^{(j,0)}(\rho)|^p \rho^{m-1-2p} d\rho\right)^{1/p} \leq C \left(\int |M(r)|^p \rho^{m-1-(1-j)p} d\rho\right)^{1/p} \leq C\|u\|_p. \quad (3.47)$$

For $I_1(x)$ we first use Hölder's inequality and then apply (3.47) to the second factor of the result to obtain

$$|I_1(x)| \leq \left(\int_{|y|\leq 1} \frac{|V\phi(x-y)|^{p'}}{|y|^{p'}} dy\right)^{1/p'} \left\| \frac{K_0^{(j,0)}(|x|)}{|x|^2} \right\|_p \leq \left(\int_{|y|\leq 1} \frac{|V\phi(x-y)|^{p'}}{|y|^{p'}} dy\right)^{1/p'} \|u\|_p,$$

where p' is the dual exponent of p and $\frac{m}{m-2} < p' \leq 2 \leq p < m/2$. Hence Minkowski's inequality implies

$$\|I_1\|_p \leq C\|u\|_p \||V\phi|^{p'}\|_{p/p'}^{1/p'} \left(\int_{|y|\leq 1} |y|^{-p'} dy\right)^{1/p'} = C\|V\phi\|_p \|u\|_p. \quad (3.48)$$

Combining (3.46), (3.47) and (3.48), we obtain the lemma also for $2 \leq p < m/2$. □

We next study $\tilde{Z}_{s1}u$ and prove the second statement of the proposition. Thus, $\phi, \psi \in \mathcal{E}$ in what follows. We begin with the case $m=3$.

Lemma 3.15. *Let $m = 3$ and $1 < p < 3$. Then, for a constant C_p independent of $u \in C_0^\infty(\mathbb{R}^3)$ we have $\|\tilde{Z}_{s1}u\|_p \le C_p\|u\|_p$.*

Proof. Define $\eta(x) = |D|^{-1}(V\psi)$. Then, by virtue of (3.14), we have that

$$\tilde{Z}_{s1}u = \int_0^\infty G_0(\lambda)|V\phi\rangle\langle\eta\,|\,(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda$$

which is the same as $\tilde{Z}_{s0}u$ with $V\psi$ being replaced by η. Thus, if we define $M(r) = M(r, \eta * \check{u})$, then the proof of (3.38), (3.40) and (3.41) implies that

$$\|\tilde{Z}_{s1}u\|_p \le C\left(\int_0^\infty |M(r)|^p r^2 dr\right)^{1/p} \le C\|\eta * u\|_p \tag{3.49}$$

when $3/2 < p < 3$ and, when $1 < p < 3/2$ that

$$\|\tilde{Z}_{s1}u\|_p \le C\left(\int_0^\infty |M(r)|^p r^{2-p} dr\right)^{1/p} \le C(\|\eta * u\|_p + \|\eta * u\|_\infty) \tag{3.50}$$

The operator $|D|^{-1}$ is the convolution with $C|x|^{-2}$ with a constant C and ψ satisfies $|V(x)\psi(x)| \le C\langle x\rangle^{-\delta-2}$ and $\int V\psi dx = 0$. It follows that $\eta(x)$ is bounded and as $|x| \to \infty$,

$$\begin{aligned}\eta(x) &= C\int\left(\frac{1}{|x-y|^2} - \frac{1}{|x^2|}\right)(V\psi)(y)dy \\ &= C\int \frac{2x\cdot y - |y|^2}{|x|^2|x-y|^2}(V\psi)(y)dy = \sum_{k=1}^3 \frac{C_{jk}x_k}{|x|^4} + O(|x|^{-4}).\end{aligned}$$

Thus $\eta \in L^q(\mathbb{R}^3)$ for any $1 < q \le \infty$ and the convolution with $\eta(x)$ is bounded in L^p for any $1 < p < \infty$ by the Calderòn-Zygmund theory (see e.g. [21], pp. 30-36). Thus, the right hand sides of (3.49) and (3.50) are both bounded by $C\|u\|_p$ for $3/2 < p < 3$ and $1 < p < 3/2$ respectively. □

Lemma 3.16. *Let $m \ge 5$ be odd and $1 < p < \frac{m}{m-1}$. Then, for a constant C_p independent of $u \in C_0^\infty(\mathbb{R}^m)$, we have $\|\tilde{Z}_{s1}u\|_p \le C\|u\|_p$.*

Proof. Here again $M(r) = M(r, (V\psi) * \check{u})$. We first prove

$$\|\tilde{Z}_{s1}^{(jk)}u\|_p \le C\|u\|_p, \quad 2 \le j \le \tfrac{m-3}{2},\ 0 \le k \le \tfrac{m-3}{2}. \tag{3.51}$$

The proof is almost a repetition of that for $m = 5$. If $2 \le j \le \frac{m-3}{2}$, then all derivatives of order up to k of $\lambda^{j+k-1}F(\lambda)\int_{\mathbb{R}} e^{-i\lambda r}r^{j+1}M(r)dr$ vanish at $\lambda = 0$ and $k+1$ integrations by parts show as in (3.43) that

$$K_1^{(j,k)}(\rho) \le_{|\cdot|} \sum_{l=0}^{k+1} \frac{C_{jkl}}{\rho^{k+1}}\mathcal{M}\mathcal{H}(r^{j+1+l}M)(\rho). \tag{3.52}$$

Since $r^{-(m-1)(p-1)}$ is an A_p weight on $\mathbb{R}$ for $1 < p < \frac{m}{m-1}$, we have

$$
\begin{aligned}
&\left(\int_0^\infty \frac{|K_1^{(j,k)}(\rho)|^p}{\rho^{(m-2-k)p}} \rho^{m-1} d\rho\right)^{1/p} \le C \sum_{l=0}^{k+1} \left(\int_0^\infty \frac{|M(r)|^p}{r^{(m-2-j-l)p}} r^{m-1} dr\right)^{1/p} \\
&\le C \left(\int_0^\infty |M(r)|^p r^{m-1} dr\right)^{1/p} + C \left(\int_0^1 \frac{|M(r)|^p}{r^{(m-4)p}} r^{m-1} dr\right)^{1/p} \\
&\le C(\|(V\psi) * u\|_p + \|(V\psi) * u\|_\infty) \le C(\|V\psi\|_1 + \|V\psi\|_{p'})\|u\|_p^p. \qquad (3.53)
\end{aligned}
$$

Here at the last step we used (3.42) and that $0 \le p(m-4) < m$. We next prove that

$$
\|(c_0 Z_{s1}^{(0,k)} - c_1 Z_{s1}^{(1,k)})u\|_p \le C\|u\|_p, \quad 0 \le k \le \tfrac{m-3}{2}. \qquad (3.54)
$$

This and (3.53) will prove the lemma. Recall $c_j = C_j|\Sigma|$, C_j are constants of (3.5) and $C_0 - iC_1 = 0$. Using (3.21), we rewrite

$$
K_1^{(0,k)}(\rho) = \frac{-i}{2\pi} \int_0^\infty e^{i\lambda\rho} \lambda^k F(\lambda) \left(\int_{\mathbb{R}} e^{-i\lambda r} \tilde{M}(r) dr\right) d\lambda. \qquad (3.55)
$$

Since the derivatives up to the order $k-1$ of $\lambda^k F(\lambda) \left(\int_{\mathbb{R}} e^{-i\lambda r} \tilde{M}(r) dr\right)$ vanishes at $\lambda = 0$ and

$$
\frac{\partial^k}{\partial \lambda^k} \left(\lambda^k F(\lambda) \int_{\mathbb{R}} e^{-i\lambda r} \tilde{M}(r) dr\right)\Big|_{\lambda=0} = k! \int_{\mathbb{R}} r^2 M(r) dr,
$$

integration by parts using $(\partial/\partial\lambda)^{k+1} e^{i\lambda\rho} = (i\rho)^{k+1} e^{i\lambda\rho}$ yields that

$$
\begin{aligned}
K_1^{(0,k)}(\rho) = \frac{i^k}{2\pi\rho^{k+1}} \Bigg(& k! \int_{\mathbb{R}} r^2 M(r) dr \\
&+ \sum_{l=0}^{k+1} \binom{k+1}{l} \int_0^\infty e^{i\lambda\rho} (\lambda^k F)^{(k+1-l)} \int_{\mathbb{R}} e^{-i\lambda r} (-ir)^l \tilde{M} dr d\lambda \Bigg). \qquad (3.56)
\end{aligned}
$$

Applying likewise integration by parts to $K_{1k,1}(\rho)$, we obtain

$$
\begin{aligned}
K_1^{(1,k)}(\rho) &= \frac{1}{2\pi} \int_0^\infty e^{i\lambda\rho} \lambda^k F(\lambda) \left(\int_{\mathbb{R}} e^{-i\lambda r} r^2 M(r) dr\right) d\lambda \\
&= \frac{i^k}{2\pi i \rho^{k+1}} \Bigg(-k! \int_{\mathbb{R}} r^2 M(r) dr \\
&- \sum_{l=0}^{k+1} \binom{k+1}{l} \int_0^\infty e^{i\lambda\rho} (\lambda^k F)^{(k+1-l)} \int_{\mathbb{R}} e^{-i\lambda r} (-ir)^l r^2 M dr d\lambda \Bigg). \qquad (3.57)
\end{aligned}
$$

Since $c_0 - ic_1 = 0$, we have by combining (3.56) and (3.57) that

$$c_0 K_1^{(0,k)}(\rho) - c_1 K_1^{(1,k)}(\rho) = \frac{c_0 i^{k-l}}{\rho^{k+1}} \sum_{l=0}^{k+1} \binom{k+1}{l} \tag{3.58}$$

$$\Big(\{\mathcal{F}^*((\lambda^k F)^{(k+1-l)}) * (\mathcal{H}(r^l \tilde{M})\}(\rho) + \{\mathcal{F}^*((\lambda^k F)^{(k+1-l)}) * \mathcal{H}(r^{l+2} M)\}(\rho) \Big)$$

$$\leq_{|\cdot|} \frac{C}{\rho^{k+1}} \sum_{l=0}^{k+1} (\mathcal{M}\mathcal{H}(r^l \tilde{M})(\rho) + \mathcal{M}\mathcal{H}(r^{l+2} M)(\rho)). \tag{3.59}$$

It follows again by using Young's inequality and the weighted inequality that

$$\begin{aligned}
&\|(c_0 Z_{s1}^{(0,k)} - c_1 Z_{s1}^{(1,k)})u\|_p \\
&\leq C \sum_{l=0}^{k+1} \|V\phi\|_1 \left(\int_0^\infty \frac{|\mathcal{M}\mathcal{H}(r^l \tilde{M}(\rho)|^p + |\mathcal{M}\mathcal{H}(r^{l+2} M(\rho)|^p}{\rho^{(m-1)p}} \rho^{m-1} d\rho \right)^{1/p} \\
&\leq C \sum_{l=0}^{k+1} \|V\phi\|_1 \left(\int_0^\infty (|\tilde{M}(r)|^p r^{pl} + |M(r)|^p r^{p(l+2)}) r^{m-1-p(m-1)} dr \right)^{1/p} \\
&\leq C \sum_{l=0}^{k+1} \|V\phi\|_1 \left(\int_0^\infty |M(r)|^p r^{p(l-m+3)+m-1} dr \right)^{1/p},
\end{aligned} \tag{3.60}$$

where we used Hardy's inequality at the last step, remembering the definition (3.20). Here $-m+3 \leq l-m+3 \leq 0$ when $0 \leq l \leq k+1$ and

$$\begin{aligned}
(3.60) &\leq C\|V\phi\|_1 \left(\int_0^1 |M(r)|^p r^{m-1-p(m-3)} dr + \int_0^\infty |M(r)|^p r^{m-1} dr \right)^{1/p} \\
&\leq C\|V\phi\|_1 (\|V\psi * u\|_\infty + \|V\psi * u\|_p) \leq C\|V\phi\|_1 (\|V\psi\|_{p'} + \|V\psi\|_p)\|u\|_p.
\end{aligned}$$

This completes the proof of the lemma. ☐

The following lemma completes the proof of Proposition 3.12.

Lemma 3.17. *Let $m \geq 5$ be odd and $\max(2, m/3) < p < m/2$. Then, for a constant C_p independent of $u \in C_0^\infty(\mathbb{R}^m)$, we have $\|\tilde{Z}_{s1} u\|_p \leq C_p \|u\|_p$.*

Proof. We prove for $0 \leq j, k \leq \frac{m-3}{2}$ that

$$\|Z_{s1}^{(j,k)} u\|_p \leq C\|u\|_p, \quad \max(2, m/3) < p < m/2. \tag{3.61}$$

Here again $M(r) = M(r, V\psi * u)$. We have

$$Z_{s1}^{(j,k)} u(x) = \left(\int_{|y|<1} + \int_{|y|\geq 1} \right) \frac{V\phi(x-y)}{|y|^{m-2-k}} K_{jk,1}(|y|) dy = I_1(x) + I_2(x). \tag{3.62}$$

Let $j \geq 1$ first. We have

$$K_1^{(j,k)}(r) = \frac{1}{2\pi}\int_0^\infty e^{i\lambda\rho}\lambda^{j+k-1}F(\lambda)\left(\frac{i\partial}{\partial\lambda}\right)^{j-1}\left(\int_{\mathbb{R}} e^{-i\lambda r}r^2M(r)dr\right)d\lambda. \tag{3.63}$$

We then apply integration by parts in a way sightly differently from that used in the proof of the previous lemma and obtain

$$\begin{aligned} K_1^{(j,k)}(r) &= \sum_{l=0}^{j-1}\binom{j-1}{l}\rho^l\{\mathcal{F}^*((\lambda^{j+k-1}F(\lambda))^{(j-1-l)}) * \mathcal{H}(r^2M)\}(\rho) \\ &\leq_{|\cdot|} \sum_{l=0}^{j-1} C_{jl}\rho^l\mathcal{M}\mathcal{H}(r^2M)(\rho) \quad j \geq 1. \end{aligned} \tag{3.64}$$

When $j=0$, we use $\tilde{M}(r)$ of (3.20) and estimate as

$$K_1^{(0,k)}(\rho) = \frac{1}{2i\pi}\int_0^\infty e^{i\lambda\rho}\lambda^kF(\lambda)\left(\int_{\mathbb{R}} e^{-i\lambda r}\tilde{M}(r)dr\right)d\lambda \leq_{|\cdot|} \mathcal{M}\mathcal{H}(\tilde{M})(\rho). \tag{3.65}$$

Let $j \geq 1$. If $\rho \geq 1$, then (3.64) is bounded by $C\rho^{j-1}\mathcal{M}\mathcal{H}(r^2M)(\rho)$ and $m-k-j-1 \geq 2$. It follows by using Young's inequality and the weighted inequality that

$$\|I_2\|_p \leq \|V\phi\|_1\left(\int_1^\infty \frac{|K_1^{(j,k)}|^p}{\rho^{p(m-2-k)}}\rho^{m-1}d\rho\right)^{1/p} \tag{3.66}$$

$$\begin{aligned} &\leq C\|V\phi\|_1\left(\int_1^\infty \frac{|\mathcal{M}\mathcal{H}(r^2M)(\rho)|^p}{\rho^{2p}}\rho^{m-1}d\rho\right)^{1/p} \\ &\leq C\|V\phi\|_1\left(\int_{\mathbb{R}}|M(r)|^pr^{m-1}dr\right)^{1/p} \leq C\|V\phi\|_1^p\|u\|_p^p, \end{aligned} \tag{3.67}$$

for $|r|^{m-1-2p}$ is an A_p weight on $\mathbb{R}$ for $m/3 < p < m/2$. When $j=0$, we estimate the right of (3.66) by first using (3.65), remarking $m-2-k \geq 2$ for $0 \leq k \leq \frac{m-3}{2}$, apply Lemma 3.1 and then Hardy's inequality consecutively. Then,

$$\begin{aligned} &\|I_2\|_p \leq \|V\phi\|_1\left(\int_1^\infty \frac{|\mathcal{M}\mathcal{H}(\tilde{M})(\rho)|^p}{\rho^{2p}}\rho^{m-1}d\rho\right)^{1/p} \\ &\leq C\|V\phi\|_1\left(\int_{\mathbb{R}}|\tilde{M}|^pr^{m-1-2p}dr\right)^{1/p} \leq C\|V\phi\|_1\left(\int_{\mathbb{R}}|M(r)|^pr^{m-1}dr\right)^{1/p}. \end{aligned}$$

The right side is bounded by $C\|V\phi\|_1\|V\psi\|_1\|u\|_p$ as previously. This and (3.67) show $\|I_2\|_p \le C\|u\|_p$. We next estimate $\|I_1\|_p$. By virtue of Hölder's inequality

$$|I_1(x)| \le \left(\int_{|y|<1}\left|\frac{V\phi(x-y)}{|y|^{m-4-k}}\right|^{p'} dy\right)^{1/p'} \left(\int_{|y|<1}\left|\frac{K_{jk}(|y|)}{|y|^2}\right|^{p} dy\right)^{1/p}. \quad (3.68)$$

When $0<\rho\le 1$, (3.64) and (3.65) imply

$$|K_1^{(j,k)}(\rho)| \le C_{jk}(\mathcal{M}\mathcal{H}(r^2M)(\rho) + \mathcal{M}\mathcal{H}(\tilde{M})(\rho))$$

where the last term is necessary only for $j=0$. Then, again by virtue of the weighted inequality and Hardy's inequality we obtain

$$\begin{aligned}\left(\int_{|y|<1}\left|\frac{K_1^{(j,k)}(|y|)}{|y|^2}\right|^{p} dy\right)^{1/p} &\le C\left(\int_{\mathbb{R}}(|M(r)r^2|^p + |\tilde{M}(r)|^p)r^{m-1-2p}d\rho\right)^{1/p} \\ &\le C\left(\int_{\mathbb{R}}|M(r)|^p r^{m-1}d\rho\right)^{1/p} \le C\|V\psi\|_1\|u\|^p. \quad (3.69)\end{aligned}$$

Since $\frac{m}{m-2} < p' < \frac{m}{m-3}$, $|x|^{-(m-4)p'}$ is integrable on $|x|\le 1$ and $p'\le p$. It follows by Minkowski's inequality

$$\left\|\left(\int_{|y|<1}\left|\frac{V\phi(x-y)}{|y|^{m-4}}\right|^{p'} dy\right)^{1/p'}\right\|_p \le C\|V\phi\|_p\left(\int_{|y|<1}\frac{dy}{|y|^{p'(m-4)}}\right)^{1/p'}.$$

Thus, $\|I_1\|_p \le C\|u\|_p$ as well. This completes the proof of the lemma. □

4 Proof of Theorem 1.1 for even $m \ge 6$

We now prove Theorem 1.1 when the space dimension $m\ge 6$ is even, viz. we prove that $Z_{\log}$ and Z_s defined respectively by (2.21) and (2.22) satisfy

$$\|Z_s u\|_p \le C\|u\|_p, \quad \|Z_{\log}u\|_p \le C\|u\|_p, \quad 1<p<m/2. \quad (4.1)$$

We prove (4.1) for Z_s first and comment on how to modify the argument for proving (4.1) for $Z_{\log}$ at the end of the section. We take the orthonormal basis $\{\phi_1,\dots,\phi_d\}$ of the 0 eigenspace $\mathcal{E}$ of H and represent $Z_s u(x) = i\pi^{-1}\sum_{l=1}^d Z_{s,l}u(x)$, where, for $l=1,\dots,d$,

$$Z_{s,l}u = \int_0^\infty G_0(\lambda)|V\phi_l\rangle\langle V\phi_l|G_0(\lambda)-G_0(-\lambda))u\rangle F(\lambda)\frac{d\lambda}{\lambda}. \quad (4.2)$$

and we prove that $Z_{s,l}u$, $l = 1, \ldots, d$ satisfies (4.1) omitting the index l in what follows. Recall $\nu = (m-2)/2$.

We want to apply the argument used for odd dimensions also to the even dimensional case as much as possible. For this purpose we represent $Z_s u(x)$ as in (4.6) below. We write $M(r) = M(r, V\phi * \check{u})$ as previously and define $Q^{a,b}_{jk}(\rho)$ for $j, k = 0, \ldots, \nu$ and $a, b > 0$ by

$$Q^{a,b}_{jk}(\rho) = \frac{(-1)^{j+1}|\Sigma|}{(1+2a)^{j+2}} \int_0^\infty \lambda^{j+k-1} e^{i\lambda(1+2b)\rho} \mathcal{F}(r^{j+1}M^a)(\lambda) F(\lambda) d\lambda, \tag{4.3}$$

where $M^a(r) = M(r/(1+2a))$. When $j = 0$, $Q^{a,b}_{0k}(\rho)$ may also be expressed as

$$Q^{a,b}_{0k}(\rho) = \frac{i|\Sigma|}{(1+2a)^2} \int_0^\infty \lambda^k e^{i\lambda(1+2b)\rho} \mathcal{F}(\widetilde{M^a})(\lambda) F(\lambda) d\lambda \tag{4.4}$$

by using (3.26). Set $\Phi_{jk}(\lambda) = \lambda^{j+k-1}F(\lambda)$. Then $\Phi_{jk} \in C_0^\infty(\mathbb{R})$ if $j + k > 0$ and,

$$\begin{aligned} Q^{a,b}_{jk}(\rho) &= (-1)^{j+1} 2\pi (1+2a)^{-(j+2)} \{(\mathcal{F}^* \Phi_{jk}) * \mathcal{H}(r^{j+1}M^a)\}((1+2b)\rho) \\ &\leq_{|\cdot|} C(1+2a)^{-(j+2)} \{\mathcal{M}\mathcal{H}(r^{j+1}M^a)\}((1+2b)\rho). \end{aligned} \tag{4.5}$$

Lemma 4.1. *Let $Q^{a,b}_{jk}(\rho)$ be as in* (4.3) *or* (4.4)*. Then,*

$$Z_s u(x) = \sum_{j,k=0}^{\nu} T_j^{(a)} T_k^{(b)} \left[\int_{\mathbb{R}^m} \frac{(V\phi)(x-y) Q^{a,b}_{jk}(|y|)}{|y|^{m-2-k}} dy \right] \equiv \sum_{j,k=0}^{\nu} Z_s^{(j,k)} u(x). \tag{4.6}$$

Proof. In the right of (4.2), use formula (3.24) for $\langle V\phi, (G_0(\lambda) - G_0(-\lambda))u \rangle$ and (3.9) for $G_0(\lambda)$, viz.

$$G_0(\lambda)\psi(x) = \sum_{k=0}^{\nu} \lambda^k T_k^{(b)} \left[\int_{\mathbb{R}^m} \frac{e^{i\lambda(1+2b)|y|}}{|y|^{m-2-k}} (V\phi)(x-y) dy \right]. \tag{4.7}$$

Then, we see that $Z_s u(x)$ is the integral with respect to $\lambda \in (0, \infty)$ of $|\Sigma|$ times

$$\sum_{j,k=0}^{\nu} T_j^{(a)} T_k^{(b)} \left[\frac{(-1)^{j+1}}{(1+2a)^{j+2}} \lambda^{j+k-1} \left(\frac{e^{i\lambda(1+2b)|y|}}{|y|^{m-2-k}} * V\phi \right) \mathcal{F}(r^{j+1}M^a)(\lambda) \right] F(\lambda).$$

Integrating with respect to λ first via Fubini's theorem yields (4.6). $\square$

4.1 Estimate of $\|Z_s^{(j,k)}u\|_p$ for $1 < p < \frac{m}{m-1}$

If $(j,k) \neq (\nu,\nu)$, the repetition of the argument modulo change of variables and the superposition via $T_j^{(a)}T_k^{(b)}$ of the previous section proves $\|Z_s^{(j,k)}u\|_p \leq C\|u\|_p$ for $1 < p < \frac{m}{m-1}$. We first check this.

4.1.1 Estimate of $\|Z_s^{(j,k)}u\|_p$, $(j,k) \neq (\nu,\nu)$ for $1 < p < \frac{m}{m-1}$.

We use the following estimate which is basically the change of variable formula. Define

$$N_\sigma^{a,b}(u) = \left(\int_0^\infty |\mathcal{M}(r^\sigma M^a)((1+2b)\rho)|^p \rho^{m-1-p(m-1)} d\rho\right)^{1/p}. \tag{4.8}$$

Lemma 4.2. *Let $0 \leq \sigma \leq m-1$ and $1 < p < m/(m-1)$. Then, for any $m/(1+\sigma) \leq q \leq \infty$ we have*

$$N_\sigma^{a,b} \leq C(1+2a)^{\frac{m}{p}-(m-1-\sigma)}(1+2b)^{m-1-\frac{m}{p}}(\|V\psi\|_1 + \|V\psi\|_q)\|u\|_p. \tag{4.9}$$

Proof. By using changes of variables and weighted inequality we have

$$\begin{aligned} N_\sigma^{a,b} &= (1+2b)^{m-1-\frac{m}{p}} \left(\int_0^\infty |\mathcal{M}(r^\sigma M^a)(\rho)|^p \rho^{m-1-p(m-1)} d\rho\right)^{1/p} \\ &\leq C(1+2a)^{\frac{m}{p}-(m-1-\sigma)}(1+2b)^{m-1-\frac{m}{p}} \left(\int_0^\infty |M(r)|^p r^{m-1-p(m-1-\sigma)} dr\right)^{1/p}. \end{aligned}$$

Denote $\kappa = m-1-\sigma$. Then $0 \leq \kappa \leq m-1$ and the last integral is bounded by

$$\begin{aligned} &\left(\int_0^\infty |M(r)|^p r^{m-1} dr\right)^{1/p} + \left(\int_0^1 |M(r)|^p r^{m-1-p\kappa} dr\right)^{1/p} \\ &\leq \|V\psi * u\|_p + \left\||x|^{-\kappa}(V\psi * u)(x)\right\|_{L^p(|x|<1)} \\ &\leq (\|V\psi\|_1 + \||x|^{-\kappa}\|_{L^{\frac{m}{\kappa}}(|x|\leq 1)}\|V\psi\|_q)\|u\|_p. \end{aligned}$$

for any $q \in [m/(1+\sigma), \infty]$ by Hölder's and weak-Young's inequality. ☐

Lemma 4.3. *Suppose $1 < p < \frac{m}{m-1}$. Then, for $2 \leq j \leq \nu$ and $0 \leq k \leq \nu$ such that $(j,k) \neq (\nu,\nu)$,*

$$\|Z_s^{(j,k)}u\|_p \leq C\|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^m). \tag{4.10}$$

Proof. Minkowski's inequality and then Young's inequality yield

$$\|Z_s^{(j,k)}u\|_p \le \|V\phi\|_1 \cdot T_j^{(a)} T_k^{(b)} \left[\left\| \frac{Q_{jk}^{a,b}(|x|)}{|x|^{m-2-k}} \right\|_p\right]. \tag{4.11}$$

We apply integration by parts $k+1$ times to (4.3) using that $e^{i\lambda(1+2b)\rho} = (i(1+2b)\rho)^{-(k+1)}\partial_\lambda^{k+1} e^{i\lambda(1+2b)\rho}$. Then, Lemma 3.3 and Lemma 3.2 imply

$$\begin{aligned} Q_{jk}^{a,b}(\rho) &= \sum_{l=0}^{k+1} \frac{(-1)^{j+1}|\Sigma|}{(1+2a)^{j+2}} \left(\frac{1}{-i(1+2b)\rho}\right)^{k+1} \binom{k+1}{l} \\ &\times \int_0^\infty e^{i\lambda(1+2b)\rho} \left(\lambda^{j+k-1}F(\lambda)\right)^{(k+1-l)} \mathcal{F}((-i)^l r^{j+l+1} M^a)(\lambda) d\lambda \qquad (4.12) \\ &\le_{|\cdot|} \sum_{l=0}^{k+1} \frac{C}{(1+2a)^{j+2}(1+2b)^{k+1}\rho^{k+1}} \mathcal{M}(r^{j+l+1}M^a)((1+2b)\rho). \qquad (4.13) \end{aligned}$$

Use Lemma 4.2 with $s \equiv j+l+1 \le m-1$ for $(j,k) \ne (\nu,\nu)$ to obtain

$$\left\| \frac{Q_{jk}^{a,b}(|x|)}{|x|^{m-2-k}} \right\|_p \le C(1+2a)^{\frac{m}{p}-(m-k-1)}(1+2b)^{m-2-\frac{m}{p}-k}\|u\|_p. \tag{4.14}$$

We then plug this to (4.11) and use $m-k-1 \ge j+2$. Then,

$$\begin{aligned} \|Z_{s2}^{(j,k)}u\|_p &\le C_{mjk} T_j^{(a)} T_k^{(b)} [(1+2a)^{\frac{m}{p}-(j+2)}(1+2b)^{m-2-\frac{m}{p}-k}]\|u\|_p \\ &\le C\|u\|_p \left(\int_0^\infty \frac{(1+2a)^{\frac{m}{p}-(j+2)}}{(1+a)^{(2\nu-j+\frac{1}{2})}} \frac{da}{\sqrt{a}}\right) \left(\int_0^\infty \frac{(1+2b)^{m-2-\frac{m}{p}-k}}{(1+b)^{(2\nu-k+\frac{1}{2})}} \frac{db}{\sqrt{b}}\right). \end{aligned}$$

Since $\frac{m}{p}-(j+2)-2\nu+j = \frac{m}{p}-m < 0$ and $m-2-\frac{m}{p} < -1$ for $1 < p < \frac{m}{m-1}$, the integrals on the right are finite and the lemma follows. □

We can repeat the argument used for odd dimensional $m \ge 5$ case also for

$$Z_{s2}^{(0,k)}u + Z_{s2}^{(1,k)}u = \int_{\mathbb{R}^m} \frac{(V\phi)(x-y)}{|y|^{m-2-k}} T_k^{(b)}(T_0^{(a)}Q_{0k}^{(a,b)}(|y|) + T_1^{(a)}Q_{0k}^{(a,b)}(|y|))dy \tag{4.15}$$

and obtain the following lemma.

Lemma 4.4. *For $1 < p < \frac{m}{m-1}$, there exists a constant $C > 0$ such that*

$$\|(Z_s^{(0,k)} + Z_s^{(1,k)})u\|_p \le C\|u\|_p, \quad k = 0, \dots, \nu. \tag{4.16}$$

Proof. We apply integration by parts $k+1$ times to (4.4) and (4.3) as in the $j \geq 2$ cases. This respectively produces the following:

$$Q_{0k}^{a,b}(\rho) = \frac{-i^k k!(\mathcal{F}\widetilde{M^a})(0)|\Sigma|}{(1+2a)^2(1+2b)^{k+1}\rho^{k+1}} - i^k|\Sigma|\sum_{l=0}^{k+1} C_{k+1,l}Q_{0k,l}^{a,b}(\rho) \tag{4.17}$$

$$Q_{1k}^{a,b}(\rho) = \frac{i^{k+1}k!\mathcal{F}(r^2M^a)(0)|\Sigma|}{(1+2a)^3(1+2b)^{k+1}\rho^{k+1}} + i^{k+1}|\Sigma|\sum_{l=0}^{k+1} C_{k+1,l}Q_{1k,l}^{a,b}(\rho), \tag{4.18}$$

where the integral terms $Q_{0k,l}^{a,b}(\rho)$ and $Q_{1k,l}^{a,b}(\rho)$ are given

$$\begin{aligned} Q_{0k,l}^{a,b}(\rho) &= \int_0^\infty \frac{e^{i\lambda(1+2b)\rho}(\lambda^k F(\lambda))^{(k+1-l)}(\mathcal{F}((-ir)^l\widetilde{M^a})(\lambda))}{(1+2a)^2(1+2b)^{k+1}\rho^{k+1}}d\lambda \\ &\leq_{|\cdot|} C\frac{\mathcal{M}(r^l\widetilde{M^a})((1+2b)\rho))}{(1+2a)^2(1+2b)^{k+1}\rho^{k+1}}. \end{aligned} \tag{4.19}$$

$$\begin{aligned} Q_{1k,l}^{a,b}(\rho) &= (-i)^l\int_0^\infty \frac{e^{i\lambda(1+2b)\rho}(\lambda^k F(\lambda))^{(k+1-l)}\mathcal{F}(r^{2+l}M^a)(\lambda))}{(1+2a)^3(1+2b)^{k+1}\rho^{k+1}}d\lambda \\ &\leq_{|\cdot|} C\frac{\mathcal{M}(r^{2+l}M^a)((1+2b)\rho)}{(1+2a)^3(1+2b)^{k+1}\rho^{k+1}}. \end{aligned} \tag{4.20}$$

We have $\mathcal{F}(\widetilde{M^a})(0) = \mathcal{F}(r^2M^a)(0) = (1+2a)^3\int_0^\infty r^2M(r)dr$ by virtue of (3.21) and, elementary computations show

$$T_1^{(a)}[i] = T_0^{(a)}[(1+2a)] = 2\nu\pi^{-\frac{1}{2}}C_m^{-1}\Gamma(1/2)\Gamma(2\nu-1).$$

It follows that the sum of the superposition via $T_0^{(a)}$ of the boundary term of (4.17) and that via $T_1^{(a)}$ of the one of (4.18) vanishes:

$$\frac{i^k k!}{(1+2b)^{k+1}\rho^{k+1}}\left(\int_0^\infty r^2M(r)dr\right)(T_1^{(a)}[i] - T_0^{(a)}[(1+2a)]) = 0. \tag{4.21}$$

For $1 < p < \frac{m}{m-1}$, $\rho^{m-1-p(m-1)}$ is an A_p weight on $\mathbb{R}$ and we have the identity:

$$\widetilde{M^a}(r) = \int_r^\infty sM^a(s)ds = (1+2a)^2\tilde{M}((1+2a)^{-1}r). \tag{4.22}$$

It follows by using Lemma 3.1 and the change of variable that

$$\left\|\frac{Q_{0k,l}^{a,b}(|x|)}{|x|^{m-k-2}}\right\|_p \leq \frac{C(1+2b)^{m-1-\frac{m}{p}}}{(1+2a)^2(1+2b)^{k+1}}\left(\int_0^\infty |r^l\widetilde{M^a}(r)|^p r^{m-1-p(m-1)}dr\right)^{1/p}$$

$$= C\frac{(1+2a)^{\frac{m}{p}-(m-1-l)}}{(1+2b)^{\frac{m}{p}-(m-k-2)}}\left(\int_0^\infty |\tilde{M}(r)|^p r^{m-1-p(m-1-l)}dr\right)^{1/p}. \tag{4.23}$$

Since $m - p(m-1-l) > 0$, Hardy's inequality applies and, because $0 \le p(m-4-k) \le p(m-3-l) < m$ for $m \ge 6$, we have as previously

$$(4.23) \le \frac{C(1+2a)^{\frac{m}{p}-(m-1-l)}}{(1+2b)^{\frac{m}{p}-(m-k-2)}}\left(\int_0^\infty |M(r)|^p r^{m-1-p(m-3-l)}dr\right)^{1/p}$$
$$\le \frac{C(1+2a)^{\frac{m}{p}-(m-k-2)}}{(1+2b)^{\frac{m}{p}-(m-k-2)}}(\|V\psi\|_1 + \|V\psi\|_{m/2})\|u\|_p,\ 0 \le l \le k+1. \tag{4.24}$$

Since $\frac{m}{p} - (m-k-2) - (2\nu+1) < -1$ and $\frac{m}{p} - (m-k-2) + 2\nu - k + 1 > 2$, $(1+2a)^{\frac{m}{p}-(m-k-2)}(1+a)^{-(2\nu+\frac{1}{2})}a^{-\frac{1}{2}}$ and $(1+2b)^{-\frac{m}{p}+(m-k-2)}(1+b)^{-(2\nu-k+\frac{1}{2})}b^{-\frac{1}{2}}$ are both integrable over $(0,\infty)$. Thus, (4.24) implies

$$T_0^{(a)}T_k^{(b)}\left[\left\|\frac{Q_{0k,l}^{a,b}(|x|)}{|x|^{m-k-2}}\right\|_p\right] \le C\|u\|_p. \quad 0 \le l \le k+1. \tag{4.25}$$

Entirely similarly, starting from (4.20), we obtain

$$\left\|\frac{Q_{1k,l}^{a,b}(|x|)}{|x|^{m-k-2}}\right\|_p \le \frac{C(1+2b)^{m-1-\frac{m}{p}}}{(1+2a)^3(1+2b)^{k+1}}\left(\int_0^\infty |r^{2+l}M^a(r)|^p r^{m-1-p(m-1)}dr\right)^{1/p}$$
$$\le \frac{C(1+2a)^{\frac{m}{p}-(m-l)}}{(1+2b)^{\frac{m}{p}-(m-k-2)}}\left(\int_0^\infty |M(r)|^p r^{m-1-p(m-3-l)}dr\right)^{1/p}$$
$$\le \frac{C(1+2a)^{\frac{m}{p}-(m-k-1)}}{(1+2b)^{\frac{m}{p}-(m-k-2)}}(\|V\psi\|_1 + \|V\psi\|_{\frac{m}{3}})\|u\|_p,\ 0 \le l \le k+1. \tag{4.26}$$

The extra decaying factor $(1+2a)^{-1}$ of (4.26) compared to (4.24) cancels the extra increasing factor $(1+a)$ of $T_1^{(a)}$ compared to $T_0^{(a)}$ and we have also

$$T_1^{(a)}T_k^{(b)}\left[\left\|\frac{Q_{0k,l}^{a,b}(|x|)}{|x|^{m-k-2}}\right\|_p\right] \le C\|u\|_p, \quad 0 \le l \le k+1. \tag{4.27}$$

Estimates (4.25) and (4.27) implies the lemma by virtue of (4.15), (4.17), (4.18) and Young's inequality as in (4.11). $\square$

4.1.2 Estimate of $Z_s^{(\nu,\nu)}$ for $1 < p < m/(m-1)$

In this subsection, we prove

$$\|Z_s^{(\nu,\nu)} u\|_p \le C\|u\|_p, \quad 1 < p < m/(m-1). \tag{4.28}$$

If we apply the method of previous section for proving this, then, in the integration by parts formula (4.12), the power $j+l+1$ of $\mathcal{F}((-i)^l r^{j+l+1})$ can reach m, which is too large to be controlled by using the weighted inequality and we have to use the different method. We first pinpoint the problem by applying the previous argument.

We apply integration by parts $\nu - 1$ times to (4.3):

$$\begin{aligned} Q_{\nu\nu}^{a,b}(\rho) &= \sum_{l=0}^{\nu-1} \frac{i^{\nu-1-l}(-1)^{\nu-1}C_{\nu-1,l}|\Sigma|}{(1+2a)^{\nu+2}(1+2b)^{\nu-1}\rho^{\nu-1}} \\ &\times \int_0^\infty e^{i(1+2b)\rho\lambda}(\lambda^{2\nu-1}F)^{(\nu-1-l)}\mathcal{F}(r^{\nu+l+1}M^a)(\lambda)d\lambda \equiv \sum_{l=0}^{\nu-1}(L_l^{a,b}u)(\rho), \end{aligned} \tag{4.29}$$

where the definition should be obvious. For $L_l^{a,b}u$ with $0 \le l \le \nu - 2$, we further apply integration by parts twice to see

$$\begin{aligned} L_l^{a,b}u(\rho) &= \sum_{j=0}^{2} C_{lj} \frac{\{\mathcal{F}^*\Phi_{\nu,j} * \mathcal{H}(r^{\nu+l+1+j}M^a)\}(\rho)}{(1+2a)^{\nu+2}(1+2b)^{\nu+1}\rho^{\nu+1}} \\ &\le_{|\cdot|} C\sum_{j=0}^{2} C_{lj} \frac{\mathcal{M}(r^{\nu+l+1+j}M^a)(\rho)}{(1+2a)^{\nu+2}(1+2b)^{\nu+1}\rho^{\nu+1}} \end{aligned} \tag{4.30}$$

Here $\nu + l + j + 1 \le 2\nu + 1 \le m - 1$. It follows by virtue of Lemma 4.2 that

$$\left\| \frac{L_l^{a,b}}{\rho^{m-2-\nu}} \right\|_p \le C_\varepsilon \frac{(1+2a)^{\frac{m}{p}}(1+2b)^{m-1-\frac{m}{p}}}{(1+2a)^{\nu+2}(1+2b)^{\nu+1}} (\|V\psi\|_1 + \|V\psi\|_{\frac{m}{\nu+2}} \|u\|_p,$$

and, as $\frac{m}{p} - (\nu+2) - \nu = m\left(\frac{1}{p} - 1\right) < 0$ and $(m-1-\frac{m}{p}) - (\nu+1) - \nu = -\frac{m}{p} < 0$,

$$\left\| T_\nu^{(a)} T_\nu^{(b)} \left[\frac{L_l}{\rho^{m-2-\nu}} \right] \right\|_p \le C(\|V\psi\|_{\frac{m}{\nu+2}} + \|V\psi\|_1)\|u\|_p, \ 0 \le l \le \nu - 2. \tag{4.31}$$

Thus, we have only to prove for $1 < p < m/(m-1)$ that

$$\left\| T_\nu^{(a)} T_\nu^{(b)} \left[\frac{L_{\nu-1}^{a,b}(\rho)}{\rho^{m-2-\nu}} \right] \right\| \le C\|u\|_p. \tag{4.32}$$

The problem here is that, if we apply integration by parts twice to obtain the factor $\rho^{-(\nu+1)}$ as in (4.30), we have $\mathcal{F}(r^{2\nu+2}M)$ there and $2\nu+2=m$ is too big to be controlled. Thus, we integrate with respect to a, b first:

$$T_\nu^{(a)}T_\nu^{(b)}\left(\frac{L_{\nu-1}^{a,b}(\rho)}{\rho^{m-\nu-2}}\right) = \frac{C}{\rho^{m-3}}\int_0^\infty \left\{ e^{i\lambda\rho}\lambda^{m-3}\left(\int_0^\infty \frac{(1+2b)^{1-\nu}}{(1+b)^{\nu+\frac{1}{2}}}e^{2i\lambda\rho b}\frac{db}{\sqrt{b}}\right)\right.$$
$$\left.\times \int_{\mathbb{R}} e^{-i\lambda r}\left(\int_0^\infty \frac{(1+2a)^{\nu-1}}{(1+a)^{\nu+\frac{1}{2}}}e^{-2ia\lambda r}\frac{da}{\sqrt{a}}\right) r^{m-2}M(r)dr\right\} F(\lambda)d\lambda. \qquad (4.33)$$

Define, for $t>0$,

$$g(t) = \int_0^\infty \left(1+\frac{b}{2t}\right)^{-\nu-\frac{1}{2}}\left(1+\frac{b}{t}\right)^{-\nu+1} e^{ib}\frac{db}{\sqrt{b}}, \qquad (4.34)$$

$$h_\pm(t) = \int_0^\infty \left(1+\frac{a}{2t}\right)^{-\nu-\frac{1}{2}}\left(1+\frac{a}{t}\right)^{\nu-1} e^{\pm ia}\frac{da}{\sqrt{a}} \qquad (4.35)$$

so that for $\lambda>0$

$$\int_0^\infty \frac{(1+2a)^{\nu-1}}{(1+a)^{\nu+\frac{1}{2}}}e^{\pm 2ia\lambda r}\frac{da}{\sqrt{a}} = \frac{1}{\sqrt{2\lambda\rho}}h_\pm(\lambda\rho), \qquad (4.36)$$

and the like for $g(\lambda\mu)$. Then, with $\chi_\pm(r)=1$ being the characteristic function of $\pm r>0$,

$$T_\nu^a T_\nu^b\left(\frac{L_{\nu-1}^{a,b}(\rho)}{\rho^{m-2-\nu}}\right) = \frac{C(m)}{2\rho^{m-\frac{5}{2}}}\int_{\mathbb{R}}(L_+(\rho,r)+L_-(\rho,r))r^{m-2}M(r)dr, \qquad (4.37)$$

$$L_\pm(\rho,r) = \frac{\chi_\pm(r)}{|r|^{\frac{1}{2}}}\int_0^\infty e^{i\lambda(\rho-r)}\lambda^{m-4}g(\lambda\rho)h_\mp(|r|\lambda)F(\lambda)d\lambda. \qquad (4.38)$$

We have the following lemma.

Lemma 4.5. *Suppose that* $f\in C^\infty([0,\infty))$ *satisfies*

$$|f^{(j)}(b)| \le C_j b^{-(j+1)} \quad j=0,1,\ldots \qquad (4.39)$$

Let

$$h_\pm(t) = \int_0^\infty e^{\pm ib}f(b/t)\frac{db}{\sqrt{b}}.$$

Then, $h_\pm(t)$ *is* C^∞ *for* $t>0$ *and satisfies the following properties.*

(1) $\tilde{h}_\pm(x)=h_\pm(1/x)$ *is of class* $C^\infty([0,1])$, *hence the limit* $\lim_{t\to\infty}h_\pm(t)=C_m$ *exists and for* $t\ge1$, $|h_\pm^{(j)}(t)|\le C_j t^{-j-1}$, $j=1,2,\ldots$.

(2) *For* $0 < t < 1$, $|t^j h_\pm^{(j)}(t)| \le C_j\sqrt{t} \le C_j$, $j = 0, 1, \dots$.

Proof. We consider $h_+(t)$ only and omit the +-sign. The proof for $h_-(t)$ is similar. It is easy to see by differentiating under the sign of integration that $h(t)$ is C^∞ for $t > 0$. We first consider the case $t > 1$. Splitting the interval of the integral, we write

$$h(t) = \left(\int_0^1 + \int_1^\infty\right) f\left(\frac{b}{t}\right) e^{ib}\frac{db}{\sqrt{b}} \equiv h_1(t) + h_2(t)$$

and $\tilde{h}_j(x) = h_j(1/x)$, $j = 0, 1$. It is obvious that $\tilde{h}_1(x)$ is C^∞ up to $x = 0$. To see the same for $\tilde{h}_2(x)$, we perform integration by parts n times. Then,

$$\begin{aligned} i^n\tilde{h}_2(x) &= \int_1^\infty f(bx)b^{-\frac{1}{2}}(e^{ib})^{(n)}db \\ &= \sum_{j=0}^{n-1}(-1)^{j+1}\partial_b^j\left(\frac{f(bx)}{\sqrt{b}}\right)(e^{ib})^{(n-j-1)}\big|_{b=1} + (-1)^n\int_1^\infty \partial_b^n\left(\frac{f(bx)}{\sqrt{b}}\right)e^{ib}db. \end{aligned} \tag{4.40}$$

Here, Leibniz's formula implies

$$\partial_b^n\left(\frac{f(bx)}{\sqrt{b}}\right) = \sum_{j=0}^{n} C_{nj}f^{(j)}(bx)(bx)^j b^{-\frac{1}{2}-n}$$

and the boundary term in (4.40)

$$\sum_{j=0}^{n-1}(-1)^{j+1}\partial_b^j\left(\frac{f(bx)}{\sqrt{b}}\right)(e^{ib})^{(n-j-1)}\big|_{b=1} = \sum_{j=0}^{n} C_{nj}f^{(j)}(x)x^j$$

is evidently C^∞. Since $\partial_y^k(f^{(j)}(y)y^j)$ is bounded for any $j, k = 0, 1, \dots$ and

$$\partial_x^k\left(\sum_{j=0}^{n} C_{nj}f^{(j)}(bx)(bx)^j b^{-\frac{1}{2}-n}\right) = \sum_{j=0}^{n} C_{nj}\ \partial_y^k(f^{(j)}(y)y^j)\big|_{y=bx} b^{-\frac{1}{2}-n+k},$$

the integral in the right of (4.40) produces a function which is of class $C^{n-1}([0,1])$. Since n is arbitrary, this proves (1). For proving (2), notice that if $\alpha(t)$ and $\beta(t)$ satisfy

$$|t^j\alpha^{(j)}(t)| \le C_j\sqrt{t}, \quad |t^j\beta^{(j)}(t)| \le C_j, \quad j = 0, 1, \dots \tag{4.41}$$

for $0 < t < 1$, then, by Leibniz' formula $\gamma(t) = \alpha(t)\beta(t)$ satisfies

$$|t^j\gamma^{(j)}(t)| \le C_j\sqrt{t}, \quad 0 < t < 1, \quad j = 0, 1, \dots. \tag{4.42}$$

It is obvious that $\sqrt{t}$ satisfies (4.42) and so does h_1 as

$$h_1(t) = \sqrt{t}\int_0^1 e^{itb} f(b)\frac{db}{\sqrt{b}} \tag{4.43}$$

is an entire function times $\sqrt{t}$.

$$h_2(t) = \sqrt{t}\int_1^\infty e^{itb} f(b)\frac{db}{\sqrt{b}} \equiv \sqrt{t}\kappa(t). \tag{4.44}$$

We show for $0 < t < 1$ that $|t^n \kappa^{(n)}(t)| \le C_n$, $n = 1, 2, \dots$. This will follow from

$$|(t^n\kappa)^{(n)}(t)| \le C_n, \quad 0 < t < 1, \quad n = 0, 1, \dots \tag{4.45}$$

by virtue of Leibniz' formula $(t^n\kappa)^{(n)} = \sum_{k=0}^n C_{nk} t^{n-k}\kappa^{(n-k)}$. By integration by parts we have

$$\begin{aligned}(it)^n\kappa(t) &= \int_1^\infty (\partial_b^n e^{itb}) f(b)\frac{db}{\sqrt{b}} \\ &= \sum_{j=0}^{n-1}(-1)^{j+1}\partial_b^j\left(\frac{f(b)}{\sqrt{b}}\right)\partial_b^{n-j-1}(e^{itb})\Big|_{b=1} + \int_1^\infty e^{itb}(f(b)b^{-\frac{1}{2}})^{(n)} db.\end{aligned}$$

The boundary term is a polynomial of t and the integral is n times continuously differentiable and a fortiori $(t^n\kappa(t))^{(n)} \le C$ for $0 < t < 1$. $\square$

Lemma 4.6. *Suppose that $g(t)$ and $h_\pm(t)$ are functions of $t > 0$ of class C^∞ and they satisfy following properties replacing f:*

(a) *The limit $\lim_{t\to\infty} f(t)$ exists.*

(b) $|t^j f^{(j)}(t)| \le C_j \begin{cases} t^{-1}, & 1 < t, \quad j = 1, 2, \dots, \\ \sqrt{t}, & 0 < t < 1, \quad j = 0, 1, \dots. \end{cases}$.

Let $\sigma \ge 0$ be an integer and let $\tilde{L}_\pm(\rho, r)$ be defined by

$$\tilde{L}_\pm(\rho, r) = \chi_\pm(r)\int_0^\infty e^{i\lambda(\rho - r)}\lambda^\sigma g(\lambda\rho)h_\mp(\pm r\lambda)F(\lambda)d\lambda \tag{4.46}$$

where $F \in C_0^\infty(\mathbb{R})$ be such that $F(\lambda) = 1$ in a neighborhood of $\lambda = 0$. Then, $\tilde{L}_\pm$ is C^∞ with respect to $\rho, r > 0$ and, for a constant $C > 0$,

$$|\tilde{L}_\pm(\rho, r)| \le C\langle \rho - r\rangle^{-(\sigma+1)} \tag{4.47}$$

Proof. We prove the lemma for $\tilde{L}_+$. The proof for $\tilde{L}_-$ is similar. It is obvious that $\tilde{L}_+(\rho, r)$ is bounded and we assume $|\rho - r| \geq 1$. We apply integration by parts $\sigma + 1$ times to

$$\tilde{L}(\rho, r) = \frac{(-i)^{\sigma+1}}{(\rho - r)^{\sigma+1}} \int_0^\infty \left(\partial_\lambda^{\sigma+1} e^{i\lambda(\rho-r)}\right) \lambda^\sigma g(\lambda\rho) h_-(r\lambda) F(\lambda) d\lambda.$$

By Leibniz' rule the derivatives of $\lambda^\sigma g(\lambda\rho)h_-(r\lambda)F(\lambda)$ of order κ is a linear combination of

$$\lambda^{\sigma-\alpha-\beta-\gamma}(\lambda\rho)^\beta g^{(\beta)}(\lambda\rho)(r^\gamma\lambda)h_-^{(\gamma)}(r\lambda)F^{(\delta)}(\lambda) \tag{4.48}$$

over $(\alpha, \beta, \gamma, \delta)$ such that $\alpha + \beta + \gamma + \delta = \kappa$ and $\alpha \leq \sigma$. This converges to 0 as $\lambda \to 0$ if $\kappa \leq \sigma$. It follows that no boundary terms appear and $(\rho - r)^{\sigma+1}\tilde{L}(\rho, r)$ is a linear combination over $(\alpha, \beta, \gamma, \delta)$ such that $\alpha + \beta + \gamma + \delta = \sigma + 1$ and $\alpha \leq \sigma$ of

$$I_{\alpha\beta\gamma\delta}(\rho, r) = \int_0^\infty e^{i(\rho-r)\lambda}\lambda^{\sigma-\alpha}\rho^\beta g^{(\beta)}(\lambda\rho)r^\gamma h_-^{(\gamma)}(r\lambda)F^{(\delta)}(\lambda)d\lambda.$$

It suffices to show that $I_{\alpha\beta\gamma\delta}(\rho, r)$ is bounded. If $\delta \neq 0$, then $c_0 < \lambda < c_1$ for constants $0 < c_0 < c_1 < \infty$ when $F^{(\delta)}(\lambda) \neq 0$, and Lemma 4.5 implies $|(\lambda\rho)^\beta g^{(\beta)}(\lambda\rho)(r\lambda)^\gamma h_-^{(\gamma)}(r\lambda)F^{(\delta)}(\lambda)| \leq C$ and

$$I_{\alpha\beta\gamma\delta}(\rho, r) \leq_{|\cdot|} C \int_{c_0}^{c_1} \lambda^{\sigma-\alpha-\beta-\gamma} d\lambda \leq C, \quad \text{if } \delta \neq 0.$$

Thus, we assume $\delta = 0$ in what follows. We may also assume $0 < r < \rho < \infty$ by symmetry. We split the region of integration into three intervals:

$$(0, \infty) = (0, 1/\rho) \cup [1/\rho, 1/r] \cup (1/r, \infty)$$

and denote the integral over these intervals by I_1, I_2 and I_3 in this order so that $I_{\alpha\beta\gamma\delta}(\rho, r) = I_1 + I_2 + I_3$.
(1) If $0 < \lambda < 1/\rho$, we have $0 < r\lambda < \rho\lambda < 1$ and

$$(\rho\lambda)^\beta g^{(\beta)}(\rho\lambda) \leq_{|\cdot|} C\sqrt{\rho\lambda}, \quad (r\lambda)^\gamma h_-^{(\gamma)}(r\lambda) \leq_{|\cdot|} C\sqrt{r\lambda}$$

It follows that

$$I_1 \leq_{|\cdot|} C \int_0^{1/\rho} \lambda^{\sigma-\alpha-\beta-\gamma}\sqrt{\rho r}\lambda d\lambda = C\sqrt{\frac{r}{\rho}} \leq C \tag{4.49}$$

(2) If $1/\rho \leq \lambda \leq 1/r$, we have $0 < r\lambda < 1 < \rho\lambda$ and we estimate as

$$(a)\ g(\rho\lambda)| \leq_{|\cdot|} C, \quad (r\lambda)^\gamma h_-^{(\gamma)}(r\lambda) \leq_{|\cdot|} C\sqrt{\lambda r} \qquad \text{if } \beta = 0,\ \gamma \neq 0.$$

(b) $(\rho\lambda)^\beta g^{(\beta)}(\rho\lambda) \le C, \quad h_-(r\lambda) \le_{|\cdot|} C\sqrt{r\lambda}, \qquad\qquad$ if $\beta \ne 0,\ \gamma = 0$.

(c) $(\rho\lambda)^\beta g^{(\beta)}(\rho\lambda) \le_{|\cdot|} C, \quad (r\lambda)^\gamma h_-^{(\gamma)}(r\lambda) \le_{|\cdot|} C\sqrt{r\lambda}, \qquad$ if $\beta \ne 0,\ \gamma \ne 0$.

In all cases, we have $\sigma - \alpha - \beta - \gamma = -1$ and

$$\lambda^{\sigma-\alpha-\beta-\gamma}(\lambda\rho)^\beta g^{(\beta)}(\lambda\rho)(r^\gamma\lambda)h_-^{(\gamma)}(r\lambda) \le_{|\cdot|} \lambda^{-\frac{1}{2}}\sqrt{r}.$$

It follows that

$$I_2 \le_{|\cdot|} C\int_{1/\rho}^{1/r} \lambda^{-\frac{1}{2}}\sqrt{r}d\lambda = 2C\sqrt{r}\left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{\rho}}\right) \le 2C. \tag{4.50}$$

(3) Finally if $1 < r\lambda < \rho\lambda$, then we likewise estimate

$$\lambda^{-1}|(\lambda\rho)^\beta g^{(\beta)}(\lambda\rho)(r^\gamma\lambda)h_-^{(\gamma)}(r\lambda)|C \begin{cases} \lambda^{-1}(r\lambda)^{-1}, & \text{if } \beta = 0,\ \gamma \ne 0 \\ \lambda^{-1}(\rho\lambda)^{-1}, & \text{if } \beta \ne 0,\ \gamma = 0, \\ \lambda^{-1}(\rho\lambda)^{-1}(r\lambda)^{-1}, & \text{if } \beta, \gamma \ne 0. \end{cases}$$

The right hand side is all bounded by $Cr^{-1}\lambda^{-2}$ and

$$I_3 \le_{|\cdot|} C\int_{1/r}^\infty \lambda^{-2}r^{-1}d\lambda = C.$$

This completes the proof. □

Now we can prove the following proposition.

Proposition 4.7. *Let $m \ge 6$. Then, for $1 \le p < m/(m-1)$, we have*

$$\|Z_s^{(\nu,\nu)}u\|_p \le C_p\|u\|_p. \tag{4.51}$$

Proof. We write $M(r)$ for $M_{V\phi*\check{u}}$ as previously. By virtue of (4.37) and (4.6), we see that $\|Z_s^{\nu\nu}u\|_p$ is bounded by a constant times

$$\sum_\pm C\|V\phi\|_1\left(\int_0^\infty \rho^{m-1}\left(\frac{1}{\rho^{m-\frac{5}{2}}}\int_{\mathbb{R}} |L_\pm(\rho,r)r^{m-2}M(r)|dr\right)^p d\rho\right)^{\frac{1}{p}}$$

and Lemma 4.6 implies that the sum of is bounded by

$$C\left(\int_0^\infty \left(\int_{\mathbb{R}} \rho^{\frac{m-1}{p}-m+\frac{5}{2}}\langle\rho - r\rangle^{-(m-3)}|r^{m-\frac{5}{2}}M(r)|dr\right)^p \rho\right)^{1/p} \tag{4.52}$$

Define $\kappa \equiv \frac{m-1}{p} - m + \frac{5}{2}$. Then, $0 < \kappa < 3/2$ for $1 < p < 2(m-1)/(2m-5)$ and

$$\rho^\kappa\langle\rho - r\rangle^{-(m-3)}|r|^{m-\frac{5}{2}} \le C\langle\rho - r\rangle^{-(m-3-\kappa)}\langle r\rangle^\kappa|r|^{m-\frac{5}{2}}$$

$$\le C \begin{cases} \langle \rho - r\rangle^{-(m-3-\kappa)}|r|^{m-\frac{5}{2}} & \text{for } |r| \le 1 \\ \langle \rho - r\rangle^{-(m-3-\kappa)}|r^{\frac{m-1}{p}}| & \text{for } |r| \ge 1. \end{cases} \tag{4.53}$$

Here $m-3-\kappa > 1$ and $\langle r\rangle^{-(m-3-\kappa)}$ is integrable on $\mathbb{R}$ if $m \ge 6$. Thus, Lemma 4.2 implies that

$$(4.53) \le C\left(\int_{|r|<1} |r^{m-\frac{5}{2}}M(r)|^p dr\right)^{\frac{1}{p}} + C\left(\int_{|r|\ge 1} |M(r)|^p|r|^{m-1}dr\right)^{\frac{1}{p}}$$
$$\le C(\|\psi\|_\sigma + \|\psi\|_1)\|u\|_p, \tag{4.54}$$

for any $\sigma > m - \frac{5}{2}$. This completes the proof. Actually, the proof shows that (4.52) holds for p such that $1 < p < 2(m-1)/(2m-5)$. $\square$

4.2 Estimate of $\|Z_s u\|_p$ for $m/3 < p < m/2$

We prove in this section that $\|Z_s u\|_p \le C\|u\|_p$ for p such that $m/3 < p < m/2$ for even $m \ge 6$. It suffices to show that $Z_s^{(j,k)}$ satisfies (4.10) for $0 \le j, k \le \nu$. We prove this by virtually repeating the argument used for proving Lemma 3.17 for odd dimensions modulo the superpositions. Let $j \ge 1$ first. Then, we split $Z_s^{(j,k)}$ as in (3.62) as

$$Z_s^{(j,k)}u(x) = T_j^{(a)}T_k^{(b)}\left[\left(\int_{|y|<1} + \int_{|y|>1}\right)\frac{(V\phi)(x-y)Q_{jk}^{a,b}(|y|)}{|y|^{m-2-k}}dy\right] \tag{4.55}$$

and estimate the integrals over $|y| \le 1$ and $|y| > 1$ separately. We write $Q_{jk}^{a,b}(\rho)$ as follows as in (3.63) and apply integration by parts in a way similar to the one used in (3.64) to see that

$$Q_{jk}^{a,b}(\rho) = |\Sigma|\int_0^\infty \frac{\lambda^{j+k-1}F(\lambda)e^{i\lambda(1+2b)\rho}\partial_\lambda^{j-1}\{\mathcal{F}(r^2M^a)(\lambda)\}}{(-i)^{j+1}(1+2a)^{j+2}}d\lambda \tag{4.56}$$
$$= \sum_{l=0}^{j-1} C_{jkl}\frac{((1+2b)\rho)^l}{(1+2a)^{j+2}}\int_0^\infty e^{i\lambda(1+2b)\rho}\Phi_{jkl}\mathcal{F}(r^2M^a)(\lambda)d\lambda$$
$$\le_{|\cdot|} \frac{C(1+2b)^{j-1}}{(1+2a)^{j+2}}\mathcal{M}\mathcal{H}(r^2M^a)((1+2b)\rho) \times \begin{cases} \rho^{j-1}, & \rho \ge 1, \\ 1, & \rho < 1, \end{cases} \tag{4.57}$$

where $\Phi_{jkl} = (\lambda^{j+k-1}F)^{(l)}$. We change variable and use Lemma 3.1 for the A_p weight ρ^{m-2p} for $m/3 < p < m/2$ to obtain

$$\left(\int_0^\infty |\{\mathcal{M}\mathcal{H}(r^2M^a)\}((1+2b)\rho)|^p.\rho^{m-1-2p}d\rho\right)^{1/p}$$

$$\le C(1+2b)^{2-\frac{m}{p}}\left(\int_0^\infty |M^a(\rho)|^p\rho^{m-1}d\rho\right)^{1/p}$$
$$\le C(1+2a)^{\frac{m}{p}}(1+2b)^{2-\frac{m}{p}}\|V\psi\|_1\|u\|_1. \tag{4.58}$$

Since $m-2-k-(j-1)\geq 2$ for $k+j\leq m-2$, it follows by using (4.57) that

$$\left\|\int_{|y|>1}\frac{(V\phi)(x-y)Q^{a,b}_{jk}(|y|)}{|y|^{m-2-k}}dy\right\| \le C\|V\phi\|_1\left(\int_1^\infty \frac{|Q^{a,b}_{jk}(\rho)|^p}{|\rho|^{p(m-2-k)}}\rho^{m-1}d\rho\right)^{1/p}$$
$$\le C\|V\phi\|_1\frac{(1+2b)^{j-1}}{(1+2a)^{j+2}}\left(\int_1^\infty |\{\mathcal{M}\mathcal{H}(r^2M^a)\}((1+2b)\rho)|^p\rho^{m-1-2p}d\rho\right)^{1/p}$$
$$\le C\|V\phi\|_1\frac{(1+2a)^{\frac{m}{p}-j-2}}{(1+2b)^{\frac{m}{p}-j-1}}\|V\psi\|_1\|u\|_p. \tag{4.59}$$

For the integral over $|y|<1$, we first estimate it via Hölder's inequality

$$\int_{|y|\le 1}\frac{|(V\phi)(x-y)Q^{a,b}_{jk}(|y|)|}{|y|^{m-k-2}}dy$$
$$\le\left(\int_{|y|\le 1}\frac{|(V\phi)(x-y)|^{p'}dy}{|y|^{p'(m-4)}}\right)^{1/p'}\left(\int_{|y|\le 1}\left|\frac{Q^{a,b}_{jk}(|y|)}{|y|^2}\right|^p dy\right)^{1/p}. \tag{4.60}$$

We then use (4.57) for the case $\rho<1$ to estimate the second factor by

$$C\frac{(1+2b)^{j-1}}{(1+2a)^{j+2}}\left(\int_0^1 |\{\mathcal{M}\mathcal{H}(r^2M^a)\}((1+2b)\rho)|^p\rho^{m-1-2p}d\rho\right)^{1/p}$$
$$\le C\frac{(1+2b)^{2-\frac{m}{p}+j-1}}{(1+2a)^{j+2-\frac{m}{p}}}\|V\psi\|_1\|u\|_p. \tag{4.61}$$

We apply Minkowski's inequality to (4.60). Since $p'\le p$ and $p'(m-4)<m$, (4.61) then implies

$$\left\|\int_{|y|\le 1}\frac{|(V\phi)(x-y)Q^{a,b}_{jk}(|y|)|}{|y|^{m-k-2}}dy\right\|_p \le C\frac{(1+2a)^{\frac{m}{p}-j-2}}{(1+2b)^{\frac{m}{p}-j-1}}\|V\psi\|_1\|V\phi\|_p\|u\|_p. \tag{4.62}$$

Combining (4.59) and (4.62) and noticing that $(1+2a)^{\frac{m}{p}-j-2}(1+2b)^{j+1-\frac{m}{p}}$ is summable with respect to $T^{(a)}_jT^{(b)}_k$, we obtain for all $1\le j\le\nu$ and $0\le k\le\nu$ that

$$\|Z^{(j,k)}_s u\|_p \le C\|V\psi\|_1(\|V\phi\|_1+\|V\phi\|_p)\|u\|_p. \tag{4.63}$$

When $j=0$, in parallel with (3.65) we have

$$Q_{0k}^{a,b}(\rho)=\frac{i|\Sigma|}{(1+2a)^2}\int_0^\infty \lambda^k e^{i\lambda(1+2b)\rho}\mathcal{F}(\widetilde{M^a})(\lambda)F(\lambda)d\lambda \tag{4.64}$$

$$\le_{|\cdot|} C(1+2a)^{-2}\mathcal{M}(\widetilde{M^a})((1+2b)\rho). \tag{4.65}$$

Then, change of variables, Lemma 3.1 for the A_p weight ρ^{m-1-2p} for $m/3<p<m/2$ and Hardy's inequality imply

$$\left(\int_0^\infty \left|\frac{Q_{0k}^{a,b}(\rho)}{\rho^2}\right|^p \rho^{m-1}d\rho\right)^{1/p} \le C\frac{(1+2b)^{2-\frac{m}{p}}}{(1+2a)^2}\left(\int_0^\infty r^{m-1-2p}|\widetilde{M^a}(r)|^p d\rho\right)^{1/p}$$
$$\le C\frac{(1+2a)^{\frac{m}{p}-2}}{(1+2b)^{\frac{m}{p}-2}}\left(\int_0^\infty r^{m-1}|M(r)|^p dr\right)^{1/p}\le C\frac{(1+2a)^{\frac{m}{p}-2}}{(1+2b)^{\frac{m}{p}-2}}\|V\psi\|_1\|u\|_p.$$

We then repeat the argument of the last part of the proof for the case $j\ge 1$, using that $m-2-k\ge 2$. This yields

$$\left\|\int_{\mathbb{R}^m}\frac{|(V\phi)(x-y)Q_{jk}^{a,b}(|y|)|}{|y|^{m-k-2}}dy\right\|_p$$
$$\le C\frac{(1+2a)^{\frac{m}{p}-2}}{(1+2b)^{\frac{m}{p}-2}}(\|V\phi\|_1+\|V\phi\|_p)\|V\psi\|_1\|u\|_p. \tag{4.66}$$

Here $\frac{m}{p}-2-2\nu<-1$. It follows that

$$\|Z_s^{0,k}u\|^p\le C(\|V\phi\|_1+\|V\phi\|_p)\|V\psi\|_1\|u\|_p, \quad k=0,\dots,\nu.$$

This completes the proof of the desired

$$\|Z_s u\|_p\le C\|u\|_p, \quad m/3<p<m/2. \tag{4.67}$$

4.3 Estimate of $Z_{\log}$

We still have to prove

$$\|Z_{\log}u\|_p\le C_p\|u\|_p, \quad 1<p<m/2, \quad m\ge 6 \text{ is even.} \tag{4.68}$$

This can be done by modifying the argument for Z_s of the previous subsections and we explain here how to do it. We do it when $m=6$ as other cases are simpler. By virtue of (2.21), it suffices to prove (4.1) for

$$Z_{\log}^{\alpha,\beta}=\frac{i}{\pi}\int_0^\infty G_0(\lambda)V\lambda^{\alpha+1}(\log\lambda)^\beta D_{\alpha,\beta}(G_0(\lambda)-G_0(-\lambda))F(\lambda)d\lambda \tag{4.69}$$

when $\alpha = 0, 1$ and $\beta = 1, 2$ and $VD_{\alpha,\beta}$ is given by (2.18). The operator with the strongest singularity with respect to λ is the one with $(\alpha, \beta) = (0, 2)$ and we deal with that operator only, omitting the suffix α, β from $Z_{\log}^{\alpha,\beta}$. Inserting (2.18) for $VD_{0,2}$ will produce $2d$ number of operators and we prove any of them satisfies (4.68), which we still denote by

$$Z_{\log} u = \frac{i}{\pi} \int_0^\infty G_0(\lambda)|\phi\rangle\langle\psi|(G_0(\lambda) - G_0(-\lambda))\lambda(\log\lambda)^2 F(\lambda) d\lambda. \tag{4.70}$$

It is important to recall that ϕ, ψ satisfy $\phi, \psi \in L^1 \cap L^6(\mathbb{R}^6)$. The argument at the beginning of Section 4 shows that $Z_{\log}$ is the sum of $Z_{\log}^{(j,k)}$, $0 \le j, k \le \nu$ which are defined by the second member of (4.6) with following changes:

(1) Change $V\phi$ in (4.6) and $V\psi$ inside $M(r)$ by ϕ and ψ of (2.18) respectively.

(2) Change λ^{j+k-1} or λ^k in the definition (4.3) or (4.4) of $Q_{jk}^{a,b}(\rho)$ by $\lambda^{j+k+1}(\log\lambda)^2$ or $\lambda^{k+2}(\log\lambda)^2$ respectively.

Then, we can check the following:

(i) Lemma 4.3 holds with $Z_{\log}^{(j,k)}$ in place of $Z_s^{(j,k)}$ for all $0 \le j, k \le \nu$ such that $(j, k) \ne (\nu, \nu)$, hence, a fortiori, so does Lemma 4.4. This is because (a) the conjugate Fourier transforms of derivatives of order up to $k + 1$ of $\lambda^{j+k+1}(\log\lambda)^2 F(\lambda)$ for $j \ge 1$ or of $\lambda^{k+2}(\log\lambda)^2 F(\lambda)$ which will appear in the integration by parts formula corresponding to (4.12) for $j \ge 1$ or (4.19) have symmetric decreasing integrable majorants, which is sufficient for obtaining (4.13) or (4.19) and because (b) $\phi, \psi \in L^1 \cap L^6(\mathbb{R}^6)$, which is sufficient to obtain Lemma 4.2 and, e.g. (4.24) and (4.26).

(2) Proposition 4.7 holds with $Z_{\log}^{(\nu,\nu)}$ in place of $Z_s^{(\nu,\nu)}$. To see this, we first remark that in the equation corresponding to (4.29) the conjugate Fourier transforms of the derivatives of $\lambda^{2\nu+1}(\log\lambda)^2 F(\lambda)$ upto the order ν have symmetric decreasing integrable majorants. This and $\phi, \psi \in L^1 \cap L^6(\mathbb{R}^6)$ implies (4.31) for the operators corresponding to $Z_{\log}$. We then need study the operator (4.33) with $\lambda^{m-1}(\log\lambda)^2$ in place of λ^{m-3} (recall we are assuming $m = 6$). By this change λ^{m-4} in (4.38) is replaced by $\lambda^{m-2}(\log\lambda)^2$. If we change λ^σ by $\lambda^\sigma(\log)^2$ in the definition (4.46) of $\tilde{L}_\pm(\rho, r)$, then the conclusion (4.47) holds with $\langle\rho - r\rangle^{-\sigma}$ in place of $\langle\rho - r\rangle^{-(\sigma+1)}$. Thus, we have in the estimates corresponding (4.52) and (4.53) the faster decaying factor $\langle\rho - r\rangle^{-(m-2)}$ in place of $\langle\rho - r\rangle^{-(m-3)}$. This produces (4.54) and we obtain Proposition 4.7 for $Z_{\log}^{(\nu,\nu)}$.

(3) With the modification as in (1) above, (4.68) for $m/3 < p < m/2$ may be proved by repeating the argument of Section 4.2 with almost no changes.

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